

# Topic 5:

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Direct Proofs of  $p \rightarrow q$

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## Handful O' Definitions (1 / 2)

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### Definition: Conjecture

### Definition: Theorem

### Definition: Proof

# Handful O' Definitions (2 / 2)

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## Definition: Lemma

## Definition: Corollary

## Example(s):

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# Why do People Fear Proofs?

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1. Proofs don't come from an assembly line.
  - ▶ Need knowledge, persistence, and creativity
  
2. Creating proofs seems like magic.
  - ▶ But they are systematic in many ways
  
3. Proofs are hard to read and understand.
  - ▶ Only if the writer makes them so
  
4. Institutionalized Fear.
  - ▶ Many teachers avoid them in classes

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# Constructing a proof? Remember:

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1. There are several proof techniques for a reason.
  - ▶ One may be easier to use than the others
2. Knowledge of mathematics is important.
  - ▶ Remember our Math Review?
3. There are “tricks” to know.
  - ▶ Ex: Dividing an even # in half leaves no remainder
4. Practice helps ... a lot!
  - ▶ Just as it does for most everything else
5. Dead ends are expected.
  - ▶ Proofs in books are the final, polished versions

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## Speaking of Even Numbers ...

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Here are a few ways to say that  $n \in \mathbb{Z}$  is even:

- $n/2$  is an integer
- $n \% 2 = 0$
- $n$  is twice some integer (e.g.,  $n = 2k, k \in \mathbb{Z}$ )

Similarly,  $d \in \mathbb{Z}$  is odd when:

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# Types Of Proof In This Class

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## 1. Direct Proof

- ▶ The most common variety

## 2. Proof by Contraposition

- ▶ Like Direct, but with a twist

## 3. Proof by Contradiction

- ▶ A dark road on a foggy night

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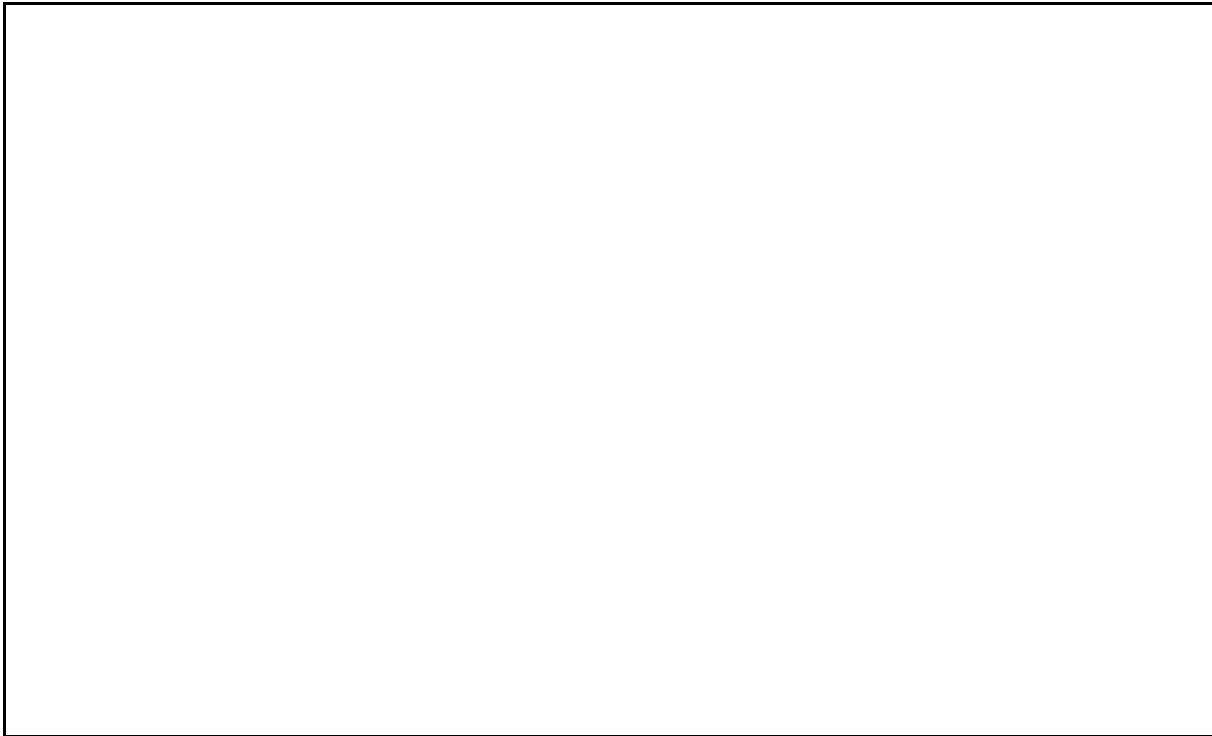
## Direct Proofs

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# Our First Conjecture

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**Conjecture:** If  $n$  is even, then  $n^2$  is also even,  $n \in \mathbb{Z}$ .



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## Proof-Writing Miscellanea

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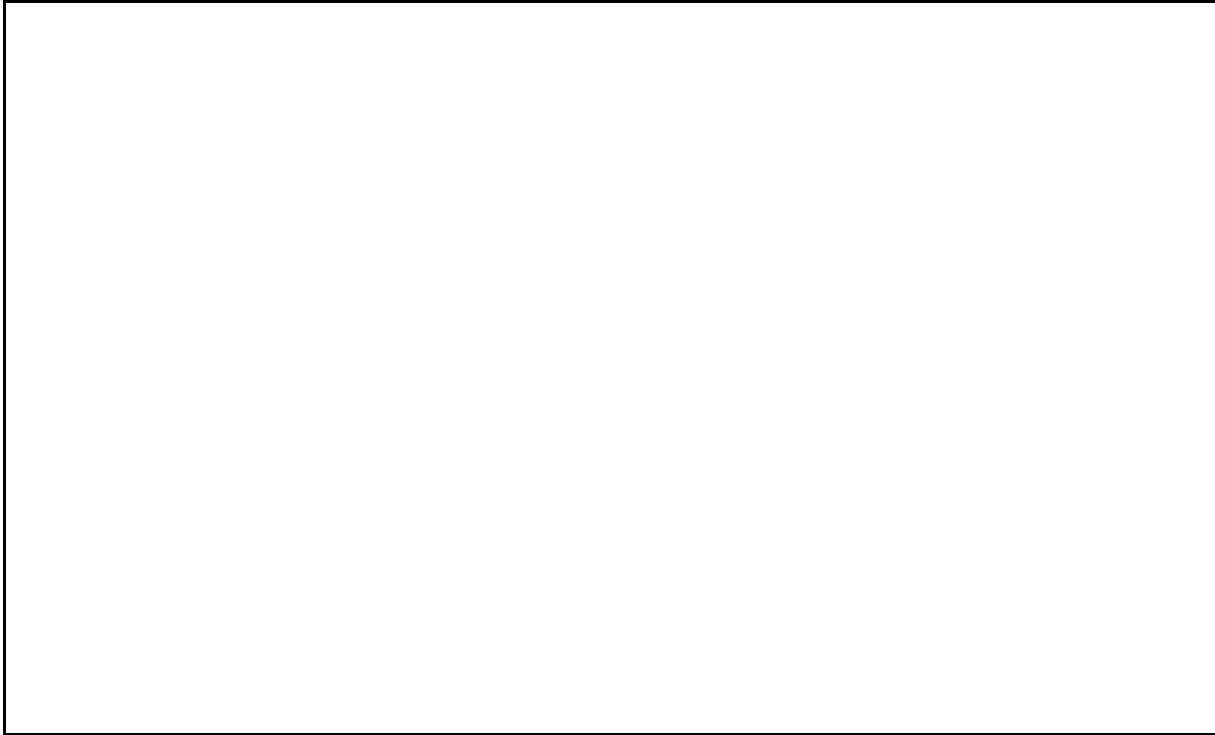
- Remember: A conjecture isn't a theorem until proven.
- Don't lose sight of your destination.
- When writing proofs in this class:
  1. Always start with "Proof (*style*):"
  2. Stating your allowed assumptions can help.
  3. Define all introduced variables.
  4. End proofs with "Therefore, " and the conjecture.

[Outside of this class: "Q.E.D." (*quod erat demonstrandum*, Latin for "this was to be demonstrated.")]

# A Conjecture About Rational Numbers

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**Conjecture:** If  $r, s \in \mathbb{Q}$ , then  $\frac{r}{s} \in \mathbb{Q}$ .

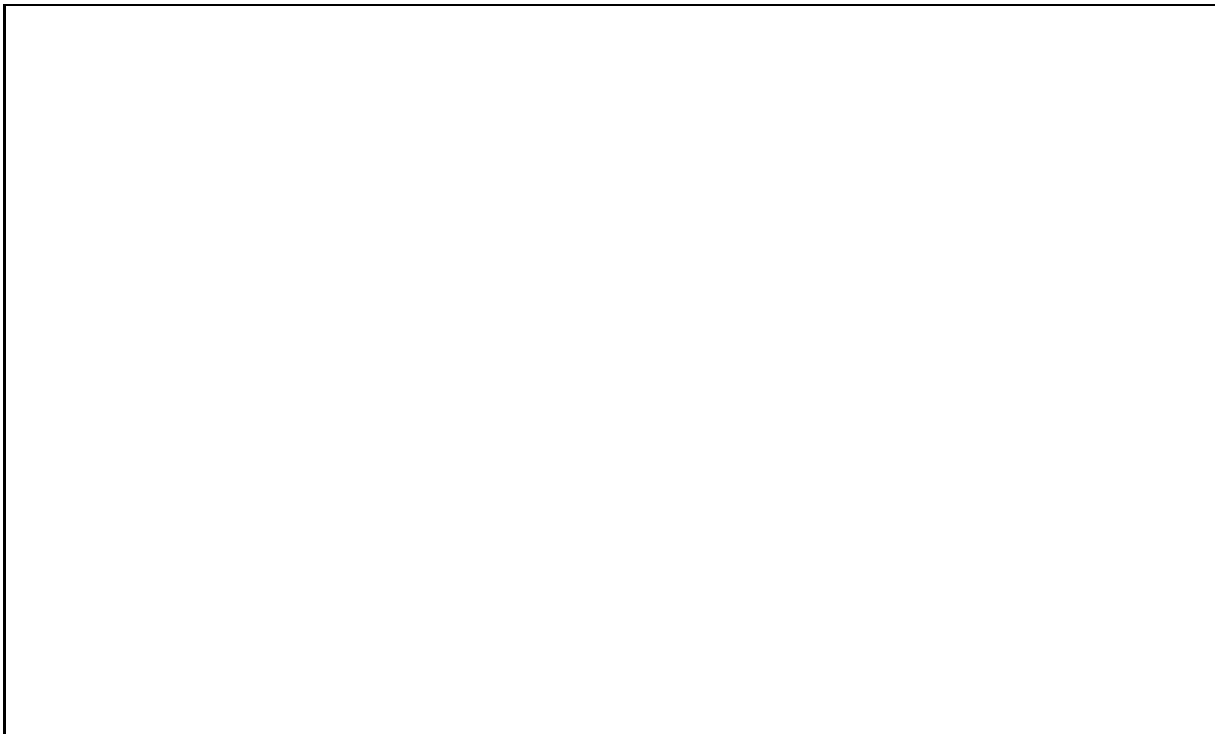


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# A Conjecture About Inequalities

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**Conjecture:** If  $0 < a < b$ , then  $a^2 < b^2$ ,  $a, b \in \mathbb{R}$ .



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# “Proof By Cases”

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**Question:** How would you prove that  $\forall x C(x)$  is true, where  $x \in \{6, 28, 496\}$ ?

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## A Direct Proof Employing Cases

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**Conjecture:**  $s \rightarrow r \equiv \neg r \rightarrow \neg s$

Proof (direct): Consider all possible combinations of values of  $r$  and  $s$ :

	$r$	$s$	$s \rightarrow r$	$\neg r \rightarrow \neg s$
Case 1:	T	T	<b>T</b>	<b>T</b>
Case 2:	T	F	<b>T</b>	<b>T</b>
Case 3:	F	T	<b>F</b>	<b>F</b>
Case 4:	F	F	<b>T</b>	<b>T</b>

Therefore,  $s \rightarrow r \equiv \neg r \rightarrow \neg s$ .

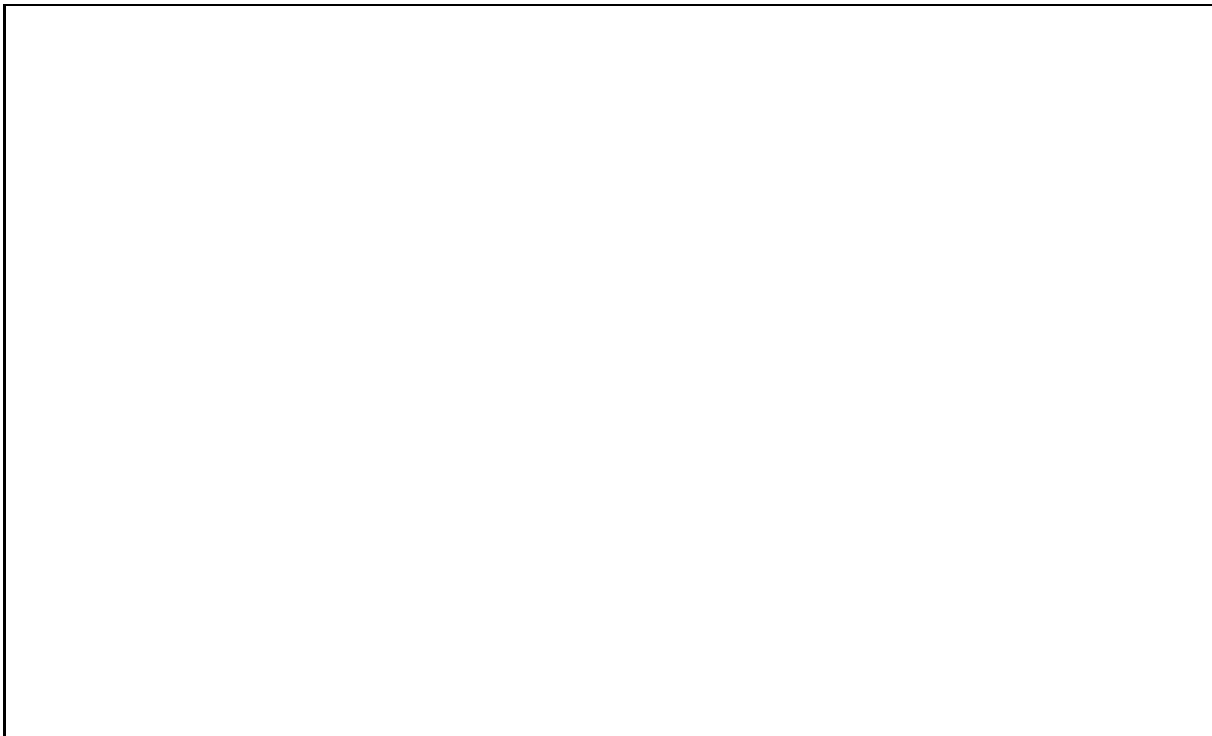
(Yes, this truth table is a direct proof by cases.)

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## A More Traditional Direct Proof With Cases

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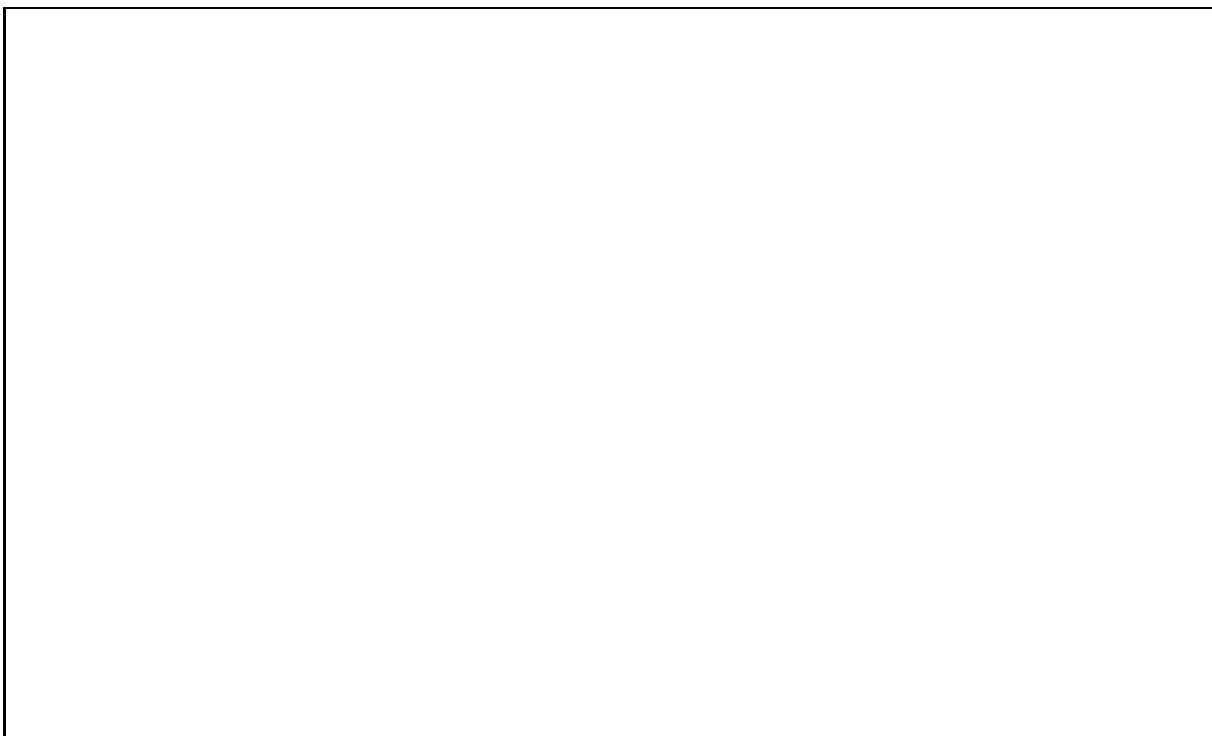
**Conjecture:**  $3x(x + 1)$  is even,  $x \in \mathbb{Z}$ .



## A More Interesting Direct Proof With Cases

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**Conjecture:**  $x^2 \% 4 \in \{0, 1\}$ ,  $x \in \mathbb{Z}$ .





## Poor Arguments $\longrightarrow$ Poor Proofs (1 / 2)

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**Conjecture:**  $1 < 0$ .

Proof or Goof?:

Consider  $x$  such that  $0 < x < 1$ . Take the base-10 logarithm of both sides of  $x < 1$ :  $\log_{10}x < \log_{10}1$ . By definition,  $\log_{10}1 = 0$ . Divide both sides by  $\log_{10}x$ :

$$\frac{\log_{10}x}{\log_{10}x} < \frac{0}{\log_{10}x}, \text{ which reduces to } 1 < 0.$$

Therefore,  $1 < 0$ .

## Poor Arguments $\longrightarrow$ Poor Proofs (2 / 2)

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**Conjecture:** For all  $n \in \mathbb{Z}^{\text{odd}}$ ,  $(n^2 - 1) \% 4 = 0$ .

Proof or Goof?:

Let  $x = 1$ .  $1^2 - 1 = 0$ .  $0 \% 4 = 0$ . Let  $x = 3$ .  $3^2 - 1 = 8$ .  $8 \% 4 = 0$ . Let  $x = 5$ .  $5^2 - 1 = 24$ .  $24 \% 4 = 0$ . This shows no sign of failing to give a result of 0.

Therefore, for all  $n \in \mathbb{Z}^{\text{odd}}$ ,  $(n^2 - 1) \% 4 = 0$ .

# Disproving Conjectures

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Direct proofs show that a conjecture is true. How can we show that a conjecture is false?

There are two disproof approaches; each is best for a specific form of conjecture.

- If the conjecture is universally-quantified (“for all”):
  
- If the conjecture is existentially-quantified (“there exists”):

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## Disproving “For All” Conjectures (1 / 2)

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Let’s start with an example from a math review concept.

**Conjecture:** Subtraction distributes over multiplication.

Put another way:  $a - (b * c) \stackrel{?}{=} (a - b) * (a - c)$ .

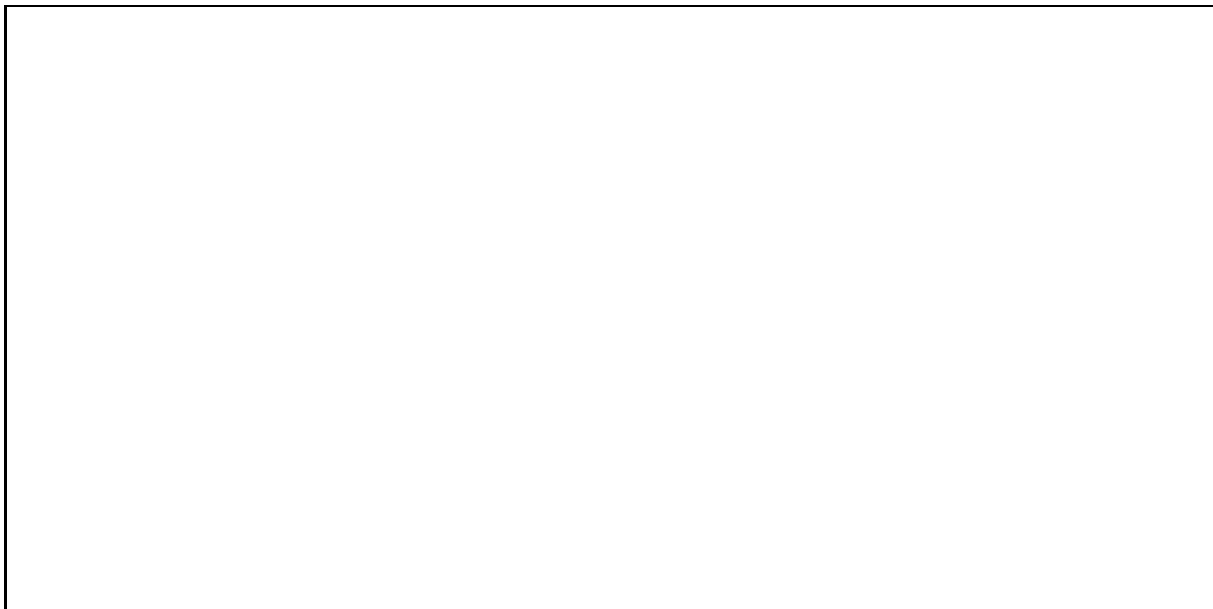
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## Disproving “For All” Conjectures (2 / 2)

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Here’s a slightly more challenging example.

**Conjecture:** No integer  $n$  exists such that the sum of its divisors equals  $2n$ .



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## Disproving “There Exists” Conjectures

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These aren’t as common, but we can handle them, too.

**Conjecture:** Someone in this room was born on Venus.

This conjecture is probably false; Venus is inhospitable!

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