

Topic 11:

Integers

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Background

In this topic we'll learn/review more properties of integer values.

We already know at least two ways in which to categorize integers:

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Prime Numbers

Definition: Factor

Definition: Prime

Definition: Composite

Example(s):

From the ‘Is This a Great Name or What?’ Dept.

Theorem: (The Fundamental Theorem of Arithmetic)

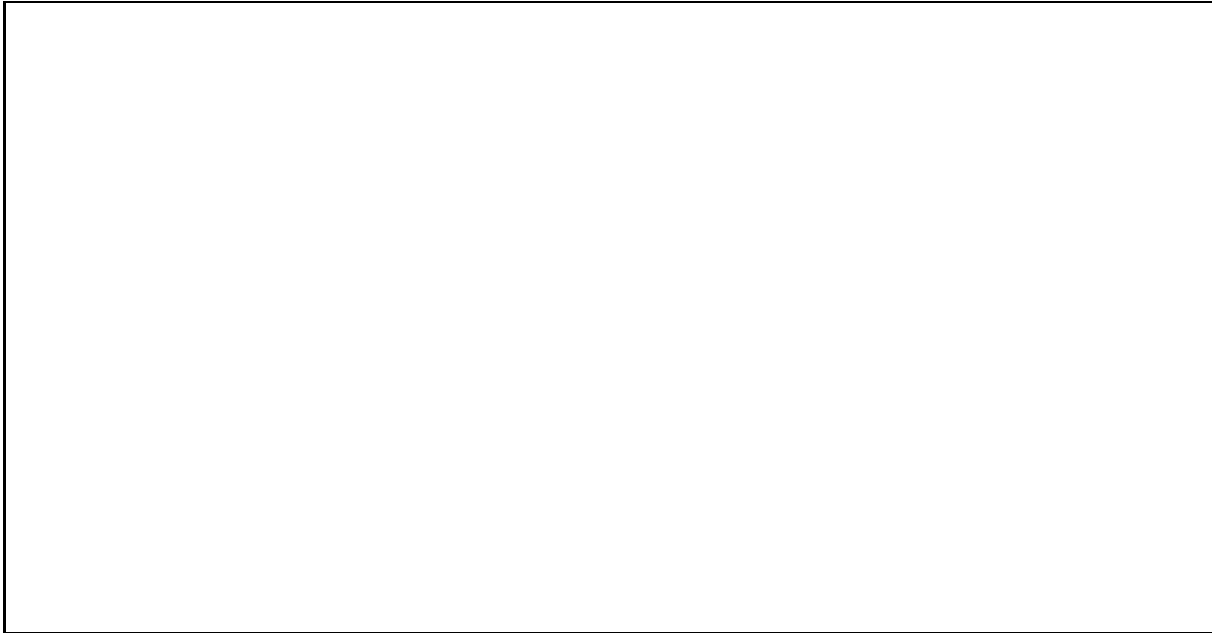
If p is a positive integer ≥ 2 , p is prime or can be expressed as the product of multiple primes.

Example(s):

Definition: Prime Factorization

Another Prime/Composite Theorem (1 / 2)

Theorem: If n is composite, n has at least one prime factor no larger than \sqrt{n} .



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Another Prime/Composite Theorem (2 / 2)

Extra room for the proof:



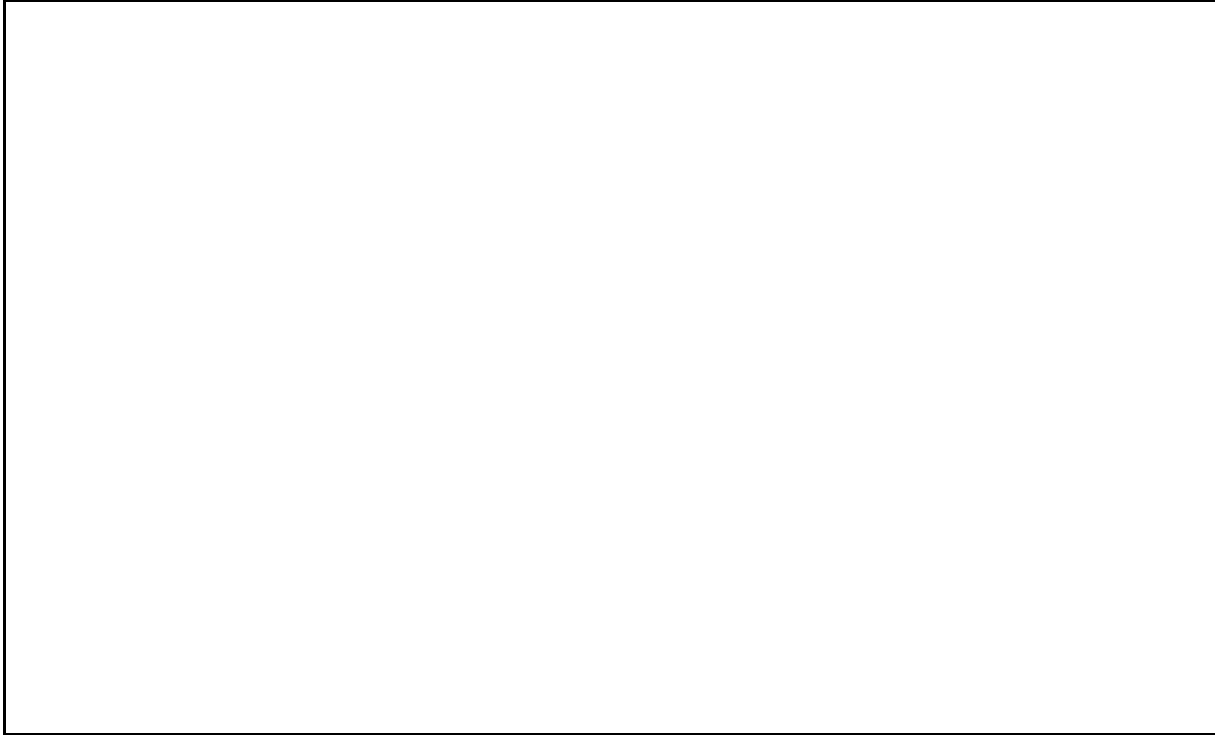
Example(s):



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Dear Euclid: How Many Primes Exist? (1 / 2)

Theorem: There are infinitely many prime integers.



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Dear Euclid: How Many Primes Exist? (2 / 2)

Useful detail: If $c \mid (a + b)$ and $c \mid a$, then $c \mid b$.

Extra room for the proof:



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Mersenne Numbers (1 / 2)

- The n^{th} Mersenne Number is $2^n - 1$ (1, 3, 7, 15, ...).
- If n is composite, $2^n - 1$ can't be prime.
 - Why not? As $n = ab$, $2^n - 1 = 2^{ab} - 1$, which is a binomial number and so it has $2^a - 1$ as a factor.
- If n is prime, $2^n - 1$ might be prime. ($2^{11} - 1 = 23 \cdot 89$)
 - If so, it's called a *Mersenne Prime*.
 - Only a few dozen such primes have been found.

So ... what's the big deal?

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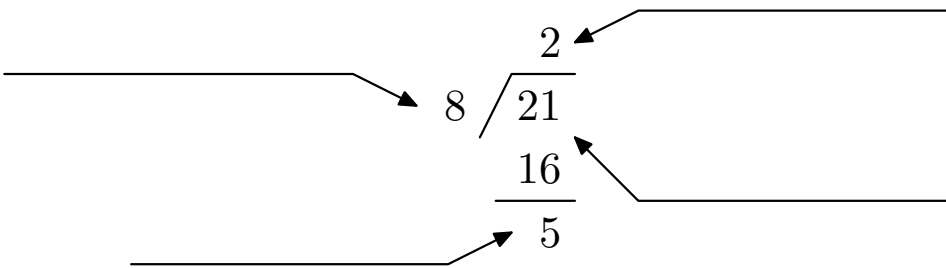
Mersenne Numbers (2 / 2)

- There exist efficient tests for the primality of Mersenne numbers (e.g., the Lucas–Lehmer test).
- The GIMPS project uses spare CPU cycles to find Mersenne primes.
 - Curious? Visit: www.mersenne.org

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Division

Name the parts!



Too Bad This Isn't Really An Algorithm

Definition: Division 'Algorithm'

Example(s):

Greatest Common Divisor (GCD) (1 / 2)

Definition: Greatest Common Divisor

.....

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Example(s):

Definition: Relatively Prime

Greatest Common Divisor (GCD) (2 / 2)

Definition: Pairwise Relatively Prime

.....

Example(s):

Least Common Multiple (LCM) (1 / 2)

Definition: Least Common Multiple

.....

.....

Example(s):

Least Common Multiple (LCM) (2 / 2)

Example(s):

At your house, the garbage is collected once a week, a new five gallon bottle of water is delivered every 10 days, and your spouse insists that you vacuum the living room every five days. Yesterday, all three occurred on the same day. How often does that happen?

Another Theorem!

Theorem: If $a, b \in \mathbb{Z}^+$, then $ab = \gcd(a, b) \cdot \text{lcm}(a, b)$.

Proof (direct): Consider the prime factorizations of a and b . The LCM is the product of the terms with the larger exponents and all terms that aren't shared. The GCD is the product of the remaining terms. Thus, the product of the LCM and GCD terms is the product of all terms in the prime factorizations.

Therefore, if $a, b \in \mathbb{Z}^+$, then $ab = \gcd(a, b) \cdot \text{lcm}(a, b)$.

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Congruences (1 / 3)

It is pitch black. You are likely to be eaten by a ...

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Congruences (2 / 3)

(Review from Topic 1.)

Definition: Congruent Modulo m

If $a, b \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, then a and b are *congruent modulo m* (written $a \equiv b \pmod{m}$) iff $a \% m = b \% m$ (or, iff $m \mid (a - b)$).

Example(s):

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Congruences (3 / 3)

Example(s):

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Number Systems: Decimal

The Base 10 (a.k.a. Decimal, a.k.a. Arabic) System

- 10 symbols (glyphs): 0,1,2,3,4,5,6,7,8 and 9.
- In a string of symbols, each position is worth the product of the symbol's value and a power of 10, starting with $10^0 = 1$ on the right.

Example(s):

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Number Systems: Binary

The Base 2 (a.k.a. Binary) System

- Just 2 symbols: 0 and 1.
- Each position is valued with increasing powers of 2.

Example(s):

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Converting Decimal to Binary

1. Repeated divisions by 2

Example(s):

2. Sums of powers of 2

Example(s):

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Number Systems: Octal

(Key point: Octal is based on groups of 3 binary digits)

The Base 8 (a.k.a. Octal) System

- 8 symbols: 0 through 7, inclusive
- Each position is valued with increasing powers of 8

Example(s):

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Converting Octal to ...

... Decimal: Multiply digits by powers of 8:

Example(s):

... Binary: Convert digits to binary, and “degroup:”

Example(s):

Number Systems: Hexadecimal

(Key point: ‘Hex’ is based on groups of 4 binary digits)

The Base 16 (a.k.a. Hexadecimal) System

- 16 symbols: 0-9, inclusive, and A-F, inclusive
- Each position is valued with increasing powers of 16

Example(s):

Why Hexadecimal Is More Common Than Octal

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Converting Between Octal and Hexadecimal

We can use decimal as a go-between, but binary is easier!

Example(s):



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Other Positional Number Systems (1 / 2)

Decimal, Binary, Octal, and Hexadecimal are the most common, but there are others! Some examples:

Base	Name	Example Use and/or Cultural Origin
3	Ternary	Teaspoon – Tablespoon – Jigger(shot)
12	Duodecimal	Dozen – Gross – Great Gross
20	Vigesimal	Basque, Celtic, Inuit, . . .
26	Hexavigesimal	Encryption using English Alphabet (glyphs A–Z)
36	Hexatrigesimal	Glyphs are 0–9 plus A–Z
60	Sexagesimal	Babylonian numerals; Seconds – Minutes – Hours

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Other Positional Number Systems (2 / 2)

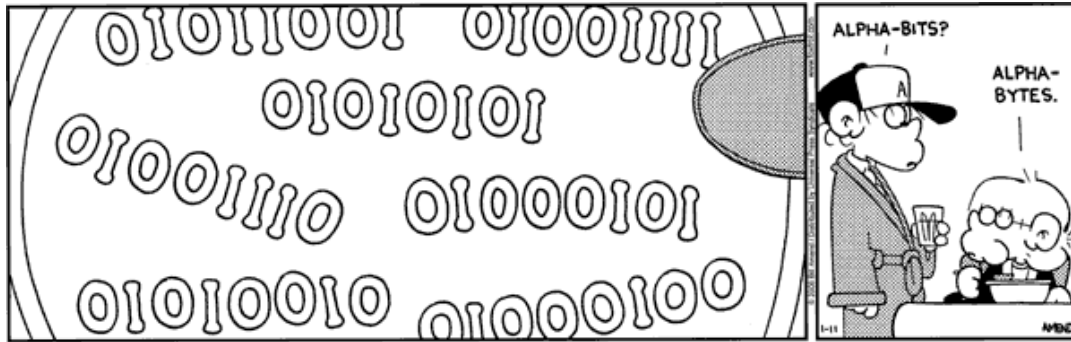
Using any positional system is easy, now that we know how they work.

Example(s):

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What's The Secret Message?

“Foxtrot” from January 11, 2006:



Hint: Find an ASCII table!