1 Master Method

Solve the following recurrences with the Master Method, if possible. Be clear to show the value of the constants $a, b$. Also identify exactly which case you are using. If a logarithm can be easily simplified (such as $\log_2 4 = 2$), do so; if not (such as $\log_5 7$), you may either convert it to a decimal value, or keep it in logarithm form.

If the recurrence cannot be solved by the Master Method, state why.

(a)
\[ T(n) = 4T\left(\frac{n}{2}\right) + n \]

**Solution:**

\[
\begin{align*}
a &= 4 \\
b &= 2 \\
\log_b a &= \log_2 4 = 2 \\
f(n) &= n \\
T(n) &= \Theta(n^2)
\end{align*}
\]

This is Case 1, because $f(n) = O(n^{2-\epsilon})$.

(b)
\[ T(n) = 3T\left(\frac{n}{3}\right) + n^3 \]

**Solution:**

\[
\begin{align*}
a &= 3 \\
b &= 3 \\
\log_b a &= \log_3 3 = 1 \\
f(n) &= n^3 \\
T(n) &= \Theta(n^3)
\end{align*}
\]

This is Case 3, because $f(n) = \Omega(n^{1+\epsilon})$. 

\[ T(n) = \Theta(n^3) \]
(c)

\[ T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} \]

**Solution:**

\[
\begin{align*}
a &= 2 \\
b &= 4 \\
\log_b a &= \log_4 2 = \frac{1}{2} \\
f(n) &= \sqrt{n}
\end{align*}
\]

This is Case 2, because \( f(n) = \Theta(n^{1/2}) \).

\[ T(n) = \Theta(\sqrt{n \log n}) \]

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(d)

\[ T(n) = 4T\left(\frac{9n}{10}\right) + n^2 \]

**Solution:**

\[
\begin{align*}
a &= 4 \\
b &= \frac{10}{9} \\
\log_b a &= \log_{10/9} 4
\end{align*}
\]

(This is something large.)

\[ f(n) = n^2 \]

(I don’t know what \( \log_{10/9} 4 \) is, but I know that it’s more than 2!)

This is Case 1, because \( f(n) = O(n^{\log_{10/9} 4 - \epsilon}) \).

\[ T(n) = \Theta(n^{\log_{10/9} 4}) \]

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(e)

\[ T(n) = 4T\left(\frac{10n}{9}\right) + n^2 \]

**Solution:**

\[
\begin{align*}
a &= 4 \\
b &= \frac{9}{10}
\end{align*}
\]

This cannot be solved by the Master Method, because \( b < 1 \).
(f) 
\[ T(n) = 2T\left(\frac{n}{2}\right) + n \lg n \]

Solution:
\[
\begin{align*}
    a &= 2 \\
    b &= 2 \\
    \log_b a &= \log_2 2 = 1 \\
    f(n) &= n \lg n \\
\end{align*}
\]

This cannot be solved by the Master Method, because \( f(n) = n \lg n \) is not \textit{polynomially} different than \( n \).

(g) 
\[ T(n) = 7T\left(\frac{n}{8}\right) + n^2 \lg n \]

Solution:
\[
\begin{align*}
    a &= 7 \\
    b &= 8 \\
    \log_b a &= \log_8 7 \\
\end{align*}
\]

(This is less than one)

\[ f(n) = n^2 \lg n \]

This is Case 3, because \( f(n) = \Omega(n^{\log_8 7 + \epsilon}) \).

\[ T(n) = \Theta(n^2 \lg n) \]

(h) 
\[ T(n) = 8T\left(\frac{n}{2}\right) + n^2 \lg n \]

Solution:
\[
\begin{align*}
    a &= 8 \\
    b &= 2 \\
    \log_b a &= \log_2 8 = 3 \\
    f(n) &= n^2 \lg n \\
\end{align*}
\]

This is Case 1, because \( f(n) = O(n^{3-\epsilon}) \).

\[ T(n) = \Theta(n^3) \]
\( T(n) = 2T\left(\frac{n}{2}\right) + \lg n \)

Solution:

\[
\begin{align*}
    a &= 2 \\
    b &= 2 \\
    \log_b a &= \log_2 2 = 1 \\
    f(n) &= \lg n
\end{align*}
\]

This is Case 1, because \( f(n) = O(n^{1-\epsilon}) \).

\( T(n) = \Theta(n) \)