Name: ____________________________ NetID: ____________________________

Person to your left: ____________________________ Person to your right: ____________________________

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1. (a) (5 points) Is the following quantification true or false? Explain your answer.

\[ \exists x \in \mathbb{R} \quad \forall y \in \mathbb{Z} \quad (x < y) \]

\[ \mathbb{R} - \text{real numbers} \]
\[ \mathbb{Z} - \text{integers} \]

(b) (5 points) In a Red-Black tree, we use widgets to simulate 2-3-4 nodes. Draw all of the possible widgets here. You don't have to give any keys for the nodes, but be very clear which nodes are black, and which are red.

(c) (5 points) What is the (normal, though not guaranteed) performance of an 'insert' operation in a hash table?

(d) (5 points) Give a recursive structural definition for a non-empty singly-linked list (as if you were about to prove something about it inductively). Don’t forget the base case!

I'll help you by starting it:

“A non-empty singly-linked list is...
(e) (5 points) Name two of the rules (out of four) of a red-black tree.

(f) (5 points) What is the best performance that is possible for any comparison sort (don’t explain why this is, simply give the performance).

What is the best performance that is possible for any type of sort? Explain your answer.

(g) (5 points) Name an example of a stable sort:

and also an unstable sort:

2. (10 points) Use radix sort to sort the following set of words. Show your work after each step!

bat
rad
bar
foo
fee
3. (15 points) Suppose that we have declared a red-black node using the following class:

```java
class RBNode {
    int key;
    RBNode left, right;
    boolean isRed;
}
```

Write a function which takes the root of a red-black tree as its parameter. It must scan the entire tree, and return the number of widgets which have exactly 3 keys in them. You may find the provided helper function useful. (You’re not required to use it, but you may.)

```java
boolean isRed(RBNode node)
{
    return (node != null) && node.isRed;
}
```

```java
int numFull(RBNode node)
{
}
```
4. (15 points) Prove the following conjecture with **structural induction**: “A nonempty ternary tree of height \( h \) has **at most** \( \frac{3^{h+1} - 1}{2} \) nodes.”

(“ternary” means 3. So this is just like a binary tree, except that each node can have up to 3 children.)
Choose only one of the induction problems to do. Do not do both - if you do, we will only grade one of them!

Prove the following conjecture using induction:

\[
\sum_{i=0}^{n} 7^i = \frac{7^{n+1} - 1}{6}
\]

for all non-negative \( n \).
5. (15 points) Each of the AVL trees below is imbalanced somewhere. Circle the node where the imbalance is detected, and then redraw the tree, to the right, to show how it would look after the required rotation(s).
6. (10 points) Run Prim’s Algorithm on the graph below, starting at node D. Show the spanning tree that you’ve selected by coloring the edges to make them much darker.

You are not required to show the contents of the priority queue as it changes - although you are welcome to do so if you want.