CSc 345 — Analysis of Discrete Structures
Fall 2024 (McCann)

http://cs.arizona.edu/classes/cs345/fall24/

Homework #1

(70 points)

Due Date: Thursday, September 5th, 2024, at the beginning of class

As we'll be seeing/writing proofs in this class, here's a homework assignment to help you awaken your possibly dormant proofing skills. There are a few questions on other topics of relevance to this class, too. We expect most are straight–forward, but one or two may challenge you. Remember that $\mathbb{Z}$ is the set of all integers, $\mathbb{Z}^+$ is the set of all positive integers, $\mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$, $\prod$ represents multiplication, $\binom{n}{k}$ is combination notation (n choose k), etc.

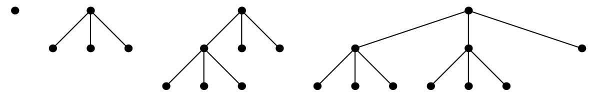
Write complete, legible answers to each of the following questions. Show your work, when appropriate, for possible partial credit. This is **NOT** a group project; do your own work. *If you need help, remember that the TAs and I (will) have office hours for just this eventuality, and you may post general questions to Piazza.*

On the due date, by the start of class, submit a PDF containing your answers on gradescope.com. Be sure to assign pages to problems after you upload your PDF. (Need help? See "Submitting an Assignment" on https://help.gradescope.com/.) If you need to submit your solutions within the 24–hour late window, Gradescope will be open to accept them. Solutions submitted after 3:30pm on the 6th will not be accepted. Want to be safe and submit your homework early? Please do! You can always resubmit an updated PDF before the due date, if necessary.

1. (4 points) How many 16-bit strings (that is, strings consisting of sixteen bits) contain exactly eight 1-bits? Explain how you arrived at your answer.
2. (4 points) In the game of Poker, a full house is a group of five cards in which three share the same rank and the other two share a different rank (e.g., 999KK). Assuming a standard deck of 52 cards (containing 4 Aces, 4 Kings, etc.), how many full house poker hands can be created? Explain how you arrived at your answer.
3. (16 points) Consider the sequence $s : 2, 8, 14, 20, 26, 32, \ldots$ Note that $s_1 = 2$.
   (a) Evaluate: $\prod_{i=1}^{3} s_{2i-1}$
   (b) Create a recurrence relation (and its initial condition) that produces s.
   (c) Construct a closed–form formula (i.e., not a recurrence, and not a summation) for the sequence s. Explain how you determined your answer.
   (d) Prove that your solution to part (c) produces the same sequence as your solution to part (b).
4. (8 points) Prove by Contradiction: If 40 coins are distributed among 9 pockets such that each pocket contains at least one coin, then at least two pockets contain the same number of coins.
5. (8 points) Prove, or disprove with a counter–example: $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.
6. (10 points) Parentheses.
   (a) How many ways are there to insert a pair of parentheses around one or more letters in a sequence of n letters? For example, if $n = 3$, there are 6 ways: $(a)bc, (ab)c, (abc), a(b)c, a(bc)$, and $ab(c)$.
   (b) Prove that your answer is correct.

(Continued...)

7. (16 points) For this problem, assume that a *complete* tree is a non–empty tree that is full of nodes on each level, excepting perhaps the last level, which is full on the left side and empty on the right. Also assume that a *complete k-ary* tree is a complete tree with the additional restriction that every internal (parent) node has exactly k children. For example, the four smallest complete 3-ary trees are shown below:



For each part below, provide an answer **and** prove its correctness.

(a) If T is a complete k-ary tree with i internal nodes, how many leaf nodes does T have?

(b) If T is a complete k-ary tree with i internal nodes, how many total nodes does T have?

8. (4 points) What is logically wrong with the following inductive "proof?"

Conjecture 1 *If a and b are positive integers, then $a = b$.*

Proof (by Induction on n, where $n = max(a, b)$):

Basis: *Let $n = 1$. If a and b are positive integers and $max(a, b) = 1$, it must be the case that $a = b = 1$.*

Inductive: *Assume that if a' and b' are positive integers and $n = max(a', b')$, then $a' = b'$. Suppose that a and b are positive integers and that $n+1 = max(a, b)$. Now $n = max(a-1, b-1)$. By the inductive hypothesis, $a - 1 = b - 1$. Thus, $a = b$.*

Therefore, if a and b are positive integers, then $a = b$. ◇