CSc 345 — Analysis of Discrete Structures
(McCann)

Problems for Practice: Recurrence Relations

Sample Problem For the following recurrence relation, find a closed–form equivalent expression and prove that it is equivalent.

$$\begin{aligned} L(1) &= 3 \\ L(n) &= L(\tfrac{n}{2}) + 1 \quad \text{where } n \text{ is a positive integral power of 2} \end{aligned}$$

Step 1: Find a closed–form equivalent expression (in this case, by use of the "Find the Pattern" approach).

$$\begin{aligned} L(1) &= 3 \\ L(n) &= L(\tfrac{n}{2}) + 1 \end{aligned}$$

$$\begin{aligned} L(\tfrac{n}{2}) &= L(\tfrac{n}{4}) + 1 \qquad \text{Express } L(n) \text{ in terms of } L(\tfrac{n}{4}) \\ L(n) &= (L(\tfrac{n}{4}) + 1) + 1 \end{aligned}$$

$$L(n) = L(\tfrac{n}{4}) + 2$$

$$\begin{aligned} L(\tfrac{n}{4}) &= L(\tfrac{n}{8}) + 1 \qquad \text{Express } L(n) \text{ in terms of } L(\tfrac{n}{8}) \\ L(n) &= (L(\tfrac{n}{8}) + 1) + 2 \end{aligned}$$

$$L(n) = L(\tfrac{n}{8}) + 3$$

$$\begin{aligned} L(\tfrac{n}{8}) &= L(\tfrac{n}{16}) + 1 \qquad \text{Once more, just to be sure of the patterns} \\ L(n) &= (L(\tfrac{n}{16}) + 1) + 3 \end{aligned}$$

$$L(n) = L(\tfrac{n}{16}) + 4$$

$$L(n) = L(\tfrac{n}{2^a}) + a \qquad \text{In general, for any postive integer } a$$

Let $2^a = n$, or $a = log_2 n$.

$$\begin{aligned} L(n) &= L(\tfrac{n}{n}) + log_2 n \\ L(n) &= L(1) + log_2 n \\ L(n) &= log_2 n + 3 \qquad \text{Final closed–form expression, which is } O(log_2 n). \end{aligned}$$

Step 2: Prove, by induction on n, that this closed–form expression is equivalent to the given recurrence relation.

<u>Theorem</u>: $L(n) = log_2 n + 3$, where n is a positive integral power of 2.

<u>Proof</u> (by induction on n):

Basis: $L(1) = log_2 1 + 3 = 0 + 3 = 3$. Correct.

Inductive: If $L(n) = log_2 n + 3$, then $L(2n) = log_2 2n + 3$

$$\begin{aligned} L(2n) &= L(n) + 1 && \text{By the given recurrence relation} \\ &= log_2 n + 3 + 1 && \text{Application of the inductive hypothesis} \\ &= (1 + log_2 n) + 3 && \text{Commutativity of Addition} \\ &= (log_2 2 + log_2 n) + 3 && log_x x = 1 \\ L(2n) &= log_2 2n + 3 && log_2 a + log_2 b = log_2(ab) \end{aligned}$$

Therefore, $L(n) = log_2 n + 3$, where n is a positive integral power of 2.

Now Try These For each of the following recurrence relations, find a closed–form equivalent expression, determine its tight $O()$ approximation, and prove that it produces the same sequence of values as does the recurrence relation.

1. $$\begin{aligned} S(0) &= 6 \\ S(n) &= S(n-1)+2 \end{aligned}$$ Easy

2. $$\begin{aligned} T(1) &= 2 \\ T(n) &= 2T(n-1)+4 \end{aligned}$$ A bit harder

3. $$\begin{aligned} Q(1) &= c \\ Q(n) &= Q(\tfrac{n}{2})+2n \end{aligned}$$ Harder still

4. For which of these can the Master Theorem be used to find a $O()$ approximation? For it/them, use the Master Theorem to confirm the $O()$ of your solution.

Looking for the solutions? I won't be posting them, because I want you to actually do them yourself (rather than just looking at the answer). If you'd like confirmation, I encourage you to share your results on Piazza, compare your work with a classmate, review them in a study group, etc.