Design Inspirations From Multivariate Polynomials, Part 1

> *Mathematics is often defined as the science of space and number. …it was not until the recent resonance of computers and mathematics that a more apt definition became fully evident: mathematics is the science of patterns.*
>
> — Lynn Arthur Steen

Ada Dietz introduced a novel method of weave design in her seminal monograph *Algebraic Expressions in Handweaving* [1]. Her idea was to use multivariate polynomials (polynomials in several variables) raised to different powers to produce sequences that could be used as the basis for design. Such design sequences can be used as profile sequences, color sequences, and so on [2-7].

Dietz Polynomials

The polynomials Ada Dietz used consist of the sum of variables with unit coefficients raised to a power. An example is $(a + b + c)^3$. **Note:** Standard mathematical notation uses italic lowercase letters at the end of the alphabet, such as x, y, and z, for variables, and roman lowercase letters at the beginning of the alphabet, such as a, b, and c for constants. Our use of letters here is deliberately different, since in many uses, variables correspond to blocks, for which the first letters of the lowercase alphabet usually are used.

The number of variables used corresponds to the number of blocks desired, while the power to which the polynomial is raised corresponds to the "degree of interaction" among the blocks.

For example, in $(a + b + c + d)^2$ there are four blocks, a, b, c, and d, with a small amount of interaction, while in $(a + b)^5$, there are two blocks, a and b, with a large amount of interaction.

Design sequences are constructed from such expressions in the following way. First, the polynomial is multiplied out, combining like terms, to give the individual terms:

1: $(a + b + c + d)^2 = a^2 + 2ab + 2ac + 2ad + b^2 + 2bc + 2bd + c^2 + 2cd + d^2$

2: $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Next, powers are replaced by products of variables:

1: $a^2 + 2ab + 2ac + 2ad + b^2 + 2bc + 2bd + c^2 + 2cd + d^2 = aa + 2ab + 2ac + 2ad + bb + 2bc + 2bd + c + 2cd + dd$

2: $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = aaaaa + 5aaab + 10aaabb + 10aabbb + 5abbbb + bbbbb$

Next, numerical coefficients are interpreted as repetitions of the variables that follow them:

1: $aa + 2ab + 2ac + 2ad + bb + 2bc + 2bd + cc + 2cd + dd \rightarrow aa + abab + acac + adad + bb + bcbc + bdbd + cc + cdcd + dd$

2: $aaaaa + 5aaab + 10aaabb + 10aabbb + 5abbbb +$
$bbbbb \rightarrow aaaaa + aaabaaabaaab + aaabbaaabbaa\_$
$abbaaabbaaabbaaabbaaabbaaabbaaabb +$
$aabbbaabbbaabbbaabbbaabbbaabbbaabbbaab\_$
$bbaabbbaabbb + abbbbabbbbabbbbabbbbabbbb +$
$bbbbb$

The underscore indicates a term that is too long to fit on the current line and is continued onto the next.

Note that this transformation produces a result that is not mathematically equivalent to the previous expression, since, for example, $abab = a^2b^2$, not $2ab$. The use of $\rightarrow$ above instead of = indicates the result is not mathematically equivalent.

Finally, the terms are concatenated to produce a profile sequence:

1: $aa + abab + acac + adad + bb + bcbc + bdbd + cc +$
$cdcd + dd \rightarrow aaababacacadadbbbcbcbdbdcc\_$
$cdcddd$

2: $aaaaa + aaabaaabaaab + aaabbaaabbaaabbaaabba\_$
$aabbaaabbaaabbaaabbaaabbaaabb + aabbbaab\_$
$bbaabbbaabbbaabbbaabbbaabbbaabbbaabbb\_$
$aabbb + abbbbabbbbabbbbabbbbabbbb +$
$bbbbb \rightarrow aaaaaaaabaaaabaaaabaaaabaaaa\_$
$baaabbaaabbaaabbaaabbaaabbaaabbaaabba\_$
$aabbaaabbaaabbaabbbaabbbaabbbabbbaabb\_$
$baabbbaabbbaabbbaabbbaabbbabbbbabbbba\_$
$bbbbabbbbabbbbbbbbb$

Although the procedure described above is not mathematically sound, it is unambiguous if somewhat arbitrary.

The actual variables used are just names and are used to stand for things like blocks and colors.

The pattern of variables in the result depends on the ordering of the variables and terms. Conventional mathematical practice is followed in the development above. Variables in the polynomials are in alphabetical order from left to right and products in terms are written in the order of the variables. Furthermore, the variables in the terms are ordered lexically (in dictionary order). For example, *aa* appears before *aab*, and *aab* appears before *aabb*. Other orderings could be used, but for uniformity, strict lexical ordering is used in the examples here.

Computing Dietz Polynomials

What is going on in deriving design sequences from polynomials is easier to see if we bypass the simplifications that usually are performed in multiplying out products of polynomials and do not use powers or combine like terms.

A simple example is $(a + b)^2$, which conventionally is multiplied out to give $a^2 + 2ab + b^2$. Instead, the multiplication process, without the use of powers and combining like terms, looks like this

$$\begin{array}{r} a + b \\ \underline{a + b} \\ ab + bb \\ \underline{aa + ab \quad\quad} \\ aa + ab + ab + bb \end{array}$$

which directly yields *aaababbb*.

So the steps in the Dietz process amount to removing simplifications usually made in polynomial arithmetic. When computing polynomial design sequences by hand, the easiest method is to avoid the simplifications usually made, going more directly to the end result (being careful to keep terms separated and in the correct order).

Design Sequence Lengths

Dietz design sequences become quite long, especially when the power ("degree of interaction") is large. Here is a table showing lengths for various numbers of variables and powers:

variables	*power*	*length*
1	1	1
1	2	2
1	3	3
1	4	4
1	5	5
1	6	6
	…	
2	1	2
2	2	8
2	3	24
2	4	64
2	5	160
2	6	384
	…	
3	1	3
3	2	18
3	3	81
3	4	324
3	5	1215
3	6	4374
	…	
4	1	4
4	2	32
4	3	192
4	4	1024
4	5	5120
4	6	24576
	…	
5	1	5
5	2	50
5	3	375
5	4	2500
5	5	15625
5	6	93750
	…	
6	1	6
6	2	72
6	3	648
6	4	5184
6	5	38880
6	6	279936
	…	
7	1	7
7	2	98
7	3	1029
7	4	9604
7	5	84035
7	6	705894
	…	
8	1	8
8	2	128
8	3	1536
8	4	16384
8	5	163840
8	6	1572864
	;	
9	1	9
9	2	162
9	3	2187

9	4	26244
9	5	295245
9	6	3188646
	…	

Sequences whose lengths are greater than several hundred are not good candidates for weave design, although parts of them may be.

Interlacement Patterns

There are many ways these sequences can be used in design, a subject we'll take up in a subsequent article.

An understanding of the nature of these sequences can be obtained by using them as threading and treadling sequences.

Interlacement patterns for patterns (drawdown images) for various Dietz polynomials are shown in the Appendix.

In these patterns, the variables *a*, *b*, *c*, … are assigned the shafts 1, 2, 3, … . Direct tie-ups are used and the treadling is as drawn in.

Note how the patterns change down the columns as the powers increase and across the rows of successive pages as the number of variables (and hence shafts and treadles) increases.

The patterns show 240 ends and picks. As the power and number of variables increase, some patterns do not show a full repeat. See the table of sequence lengths given on the previous page.

More to Come

In the next article on polynomials as a basis for design, we'll look at extensions to Ada Dietz's methods. And in another article, we'll explore the design possibilities based on polynomials.

References

1. *Algebraic Expressions in Handwoven Textiles*, Ada K. Dietz, The Little Loomhouse, Louisville, Kentucky, 1949.

2. "Algebraic Formulae for Draft Writing", Nellie Sargent Johnson, *Handweaving News*, July 1947.

3. "Two Weavers in a Trailer", *Handweaver & Craftsman*, Spring 1953, pp. 20-21, 56, 60.

4. "Variations on an Algebraic Equation", Gail Redfield, *Handweaver & Craftsman*, Summer 1959, pp. 46-49.

5. *Algebraic Expressions in Design: A Study Guide*, Eileen Hallman, self published, 1997.

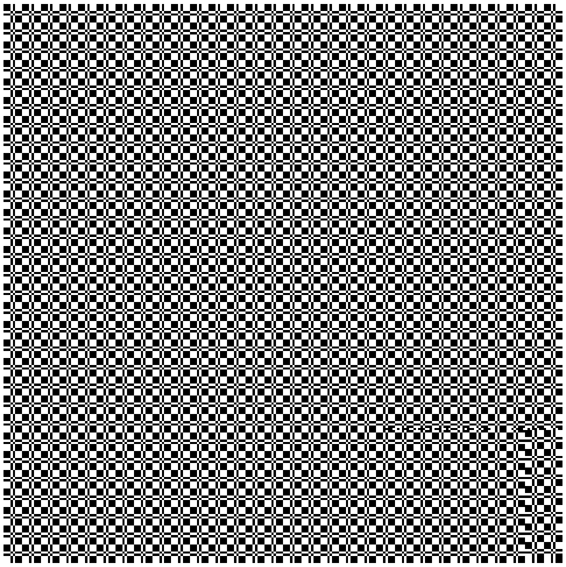
6. "Algebraic Expressions: Designs for Weaving", Lana Schneider, *Handwoven*, January/February 1998, pp. 48-49.

7. *Algebraic Expressions as Design Elements*, Lana Schneider, http://anwg.org/resources/articles/algebra.html

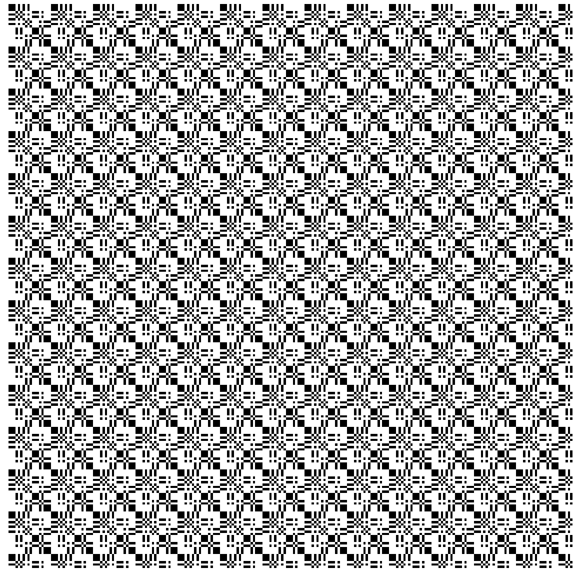
Ralph E. Griswold

Department of Computer Science
The University of Arizona, Tucson, Arizona

Appendix



$(a + b)^2$



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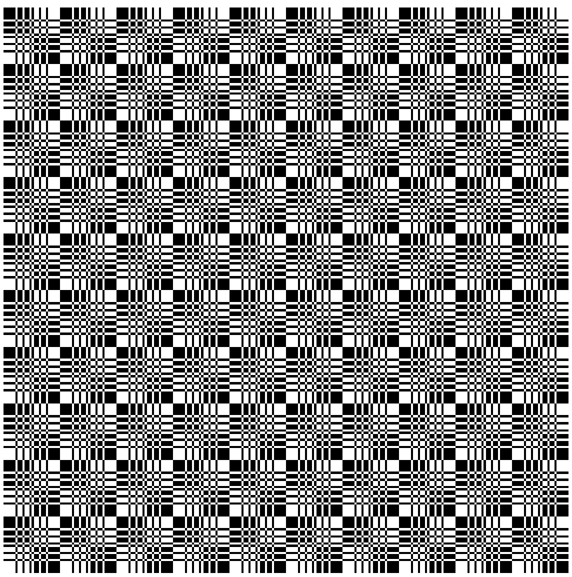
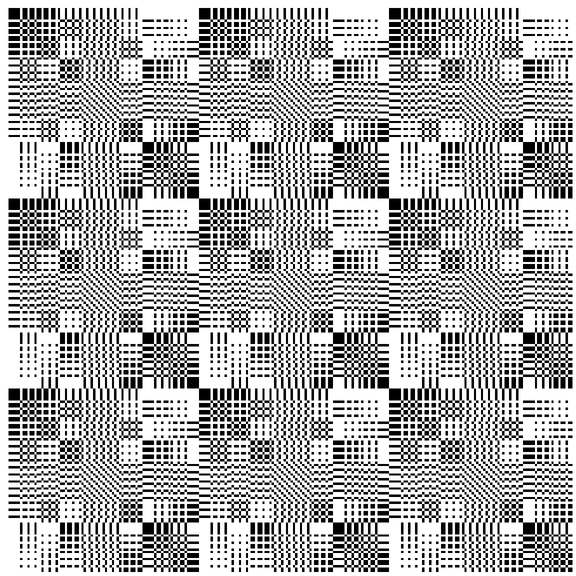


$(a + b)^3$



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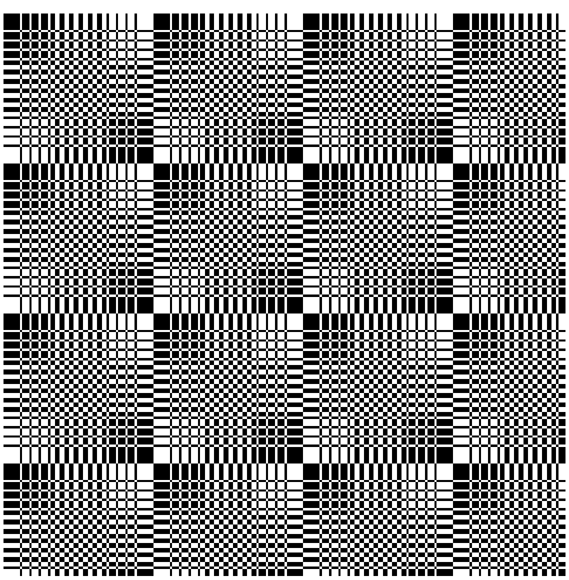

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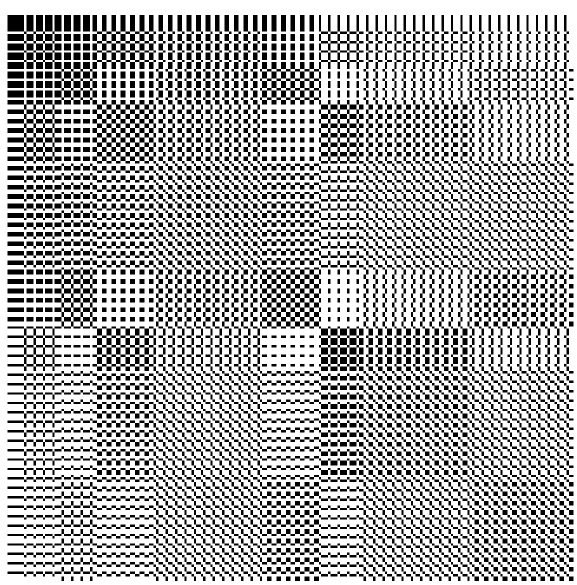
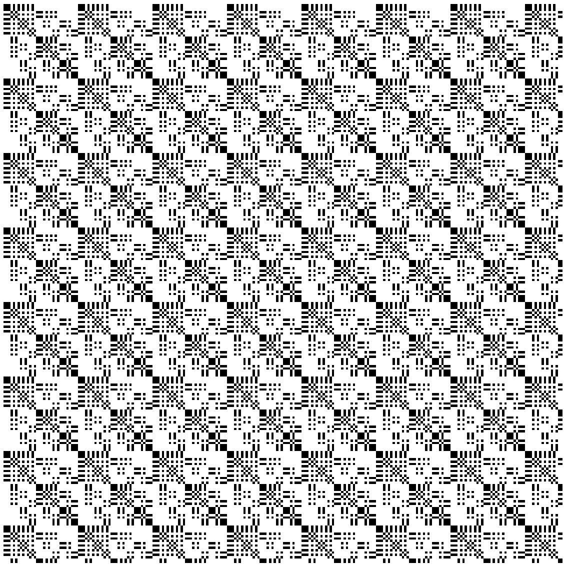
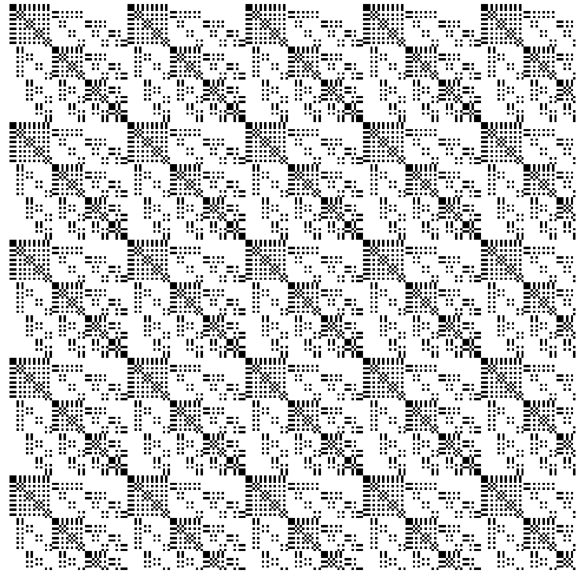


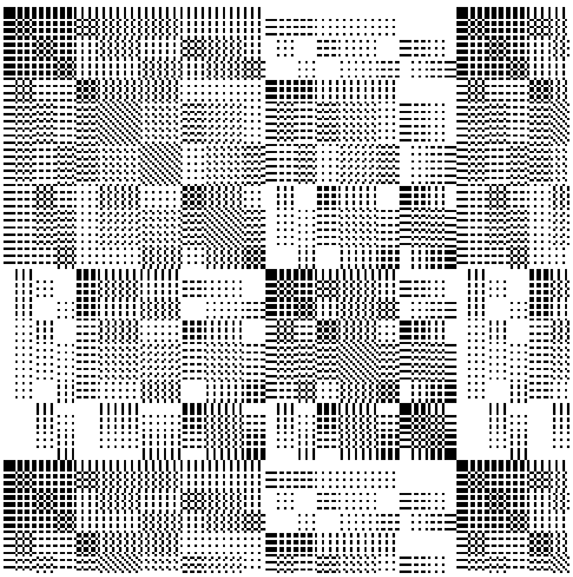
$(a + b + c)^4$



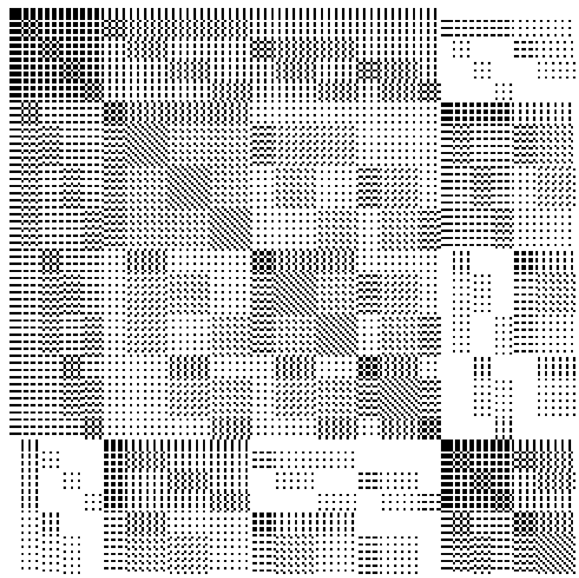
(*a* + *b* + *c* + *d*)²



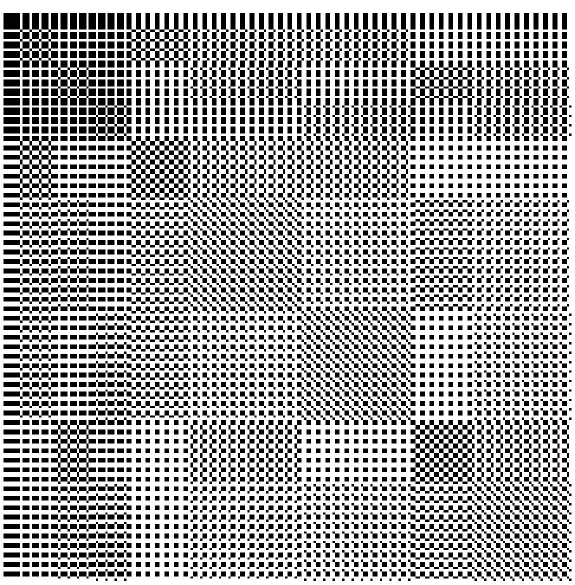
(*a* + *b* + *c* + *d* + *e*)²



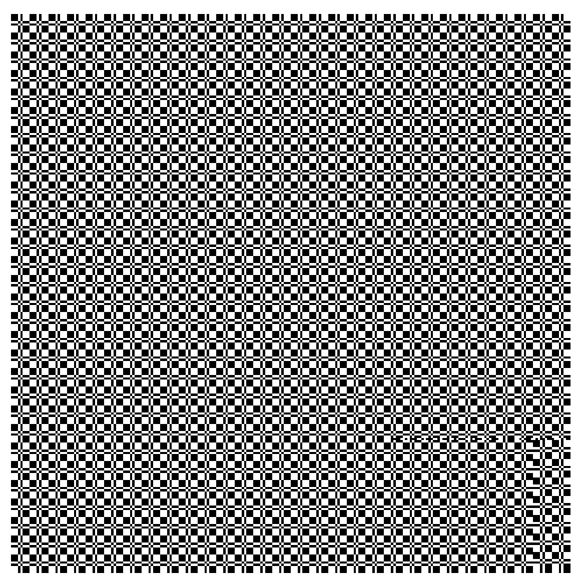
(*a* + *b* + *c* + *d*)³



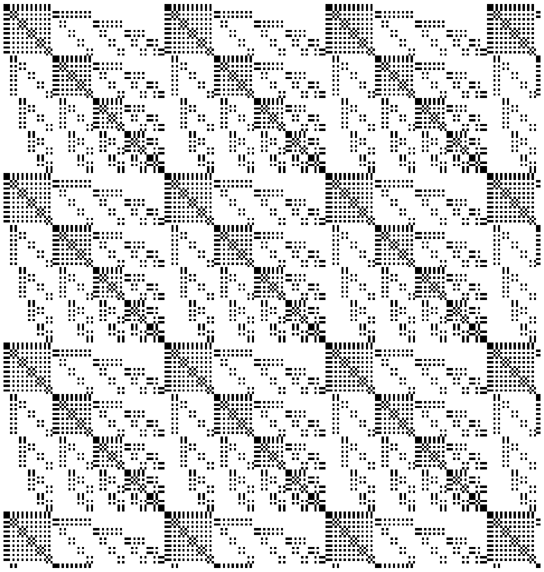
(*a* + *b* + *c* + *d* + *e*)³



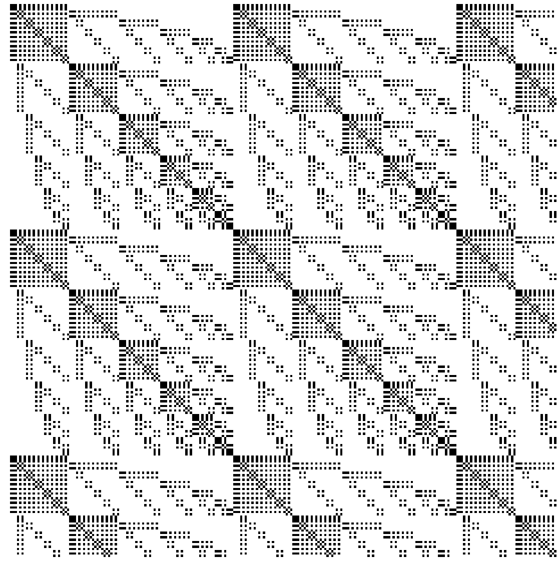
(*a* + *b* + *c* + *d*)⁴



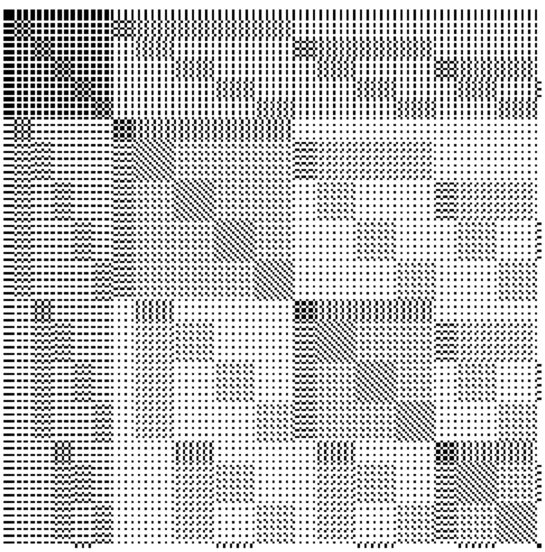
(*a* + *b* + *c* + *d* + *e*)⁴



$(a + b + c + d + e + f)^2$



$(a + b + c + d + e + f + g)^2$



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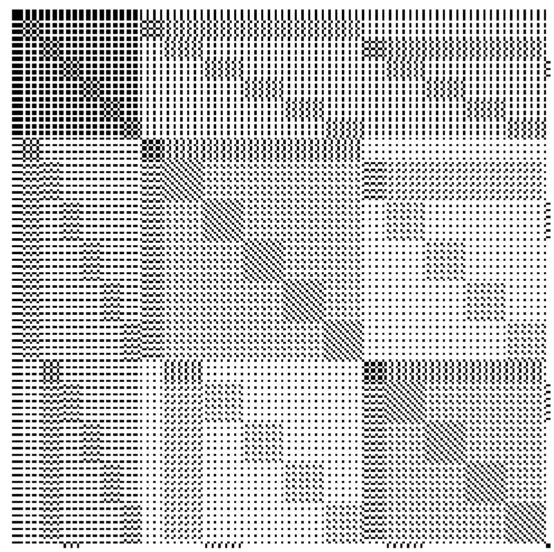
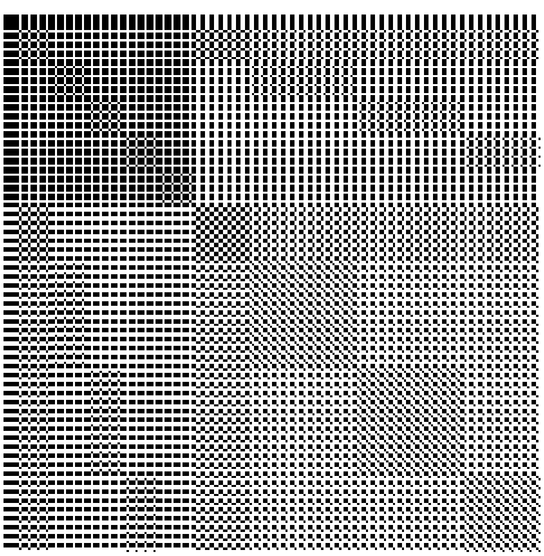
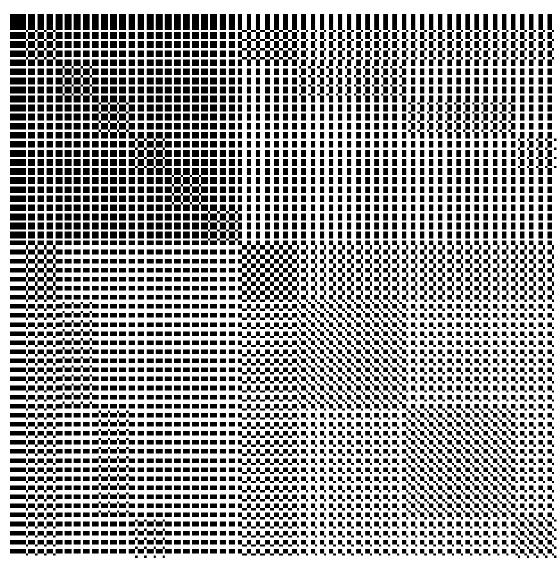


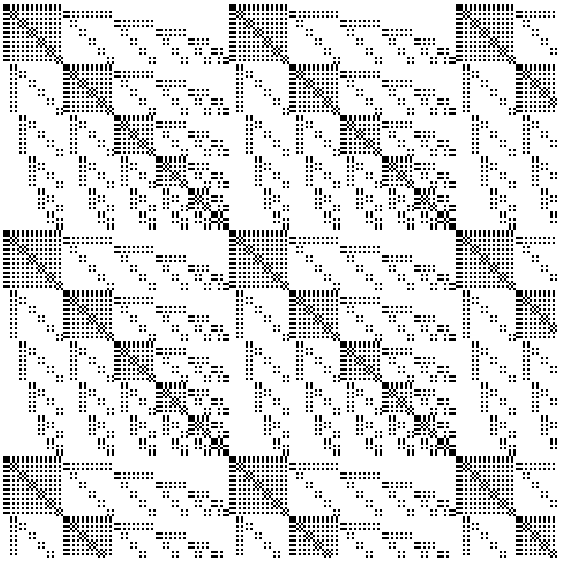
$(a + b + c + d + e + f + g)^3$



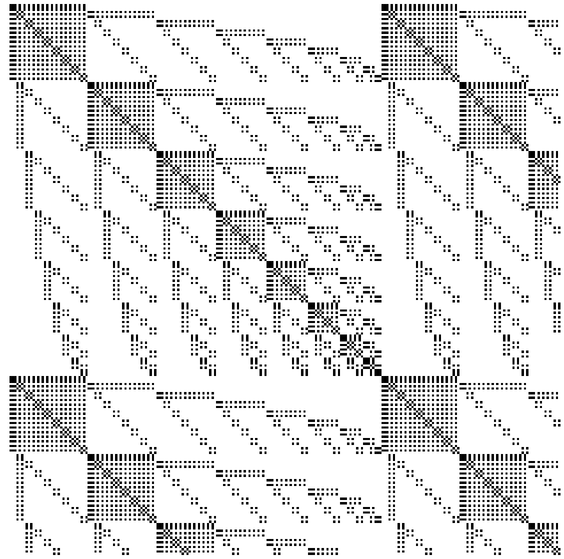
$(a + b + c + d + e + f)^4$



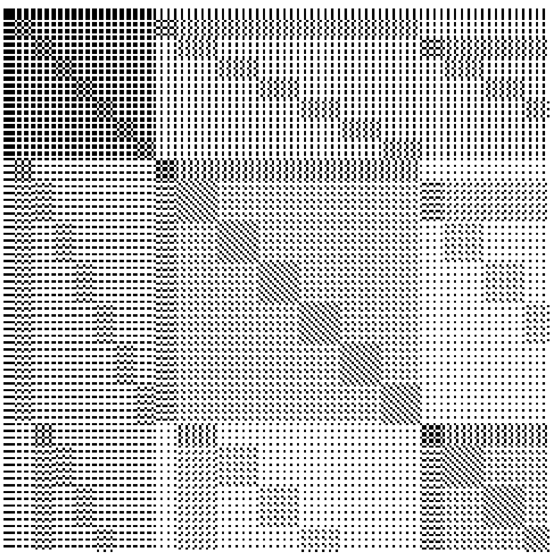
$(a + b + c + d + e + f + g)^4$



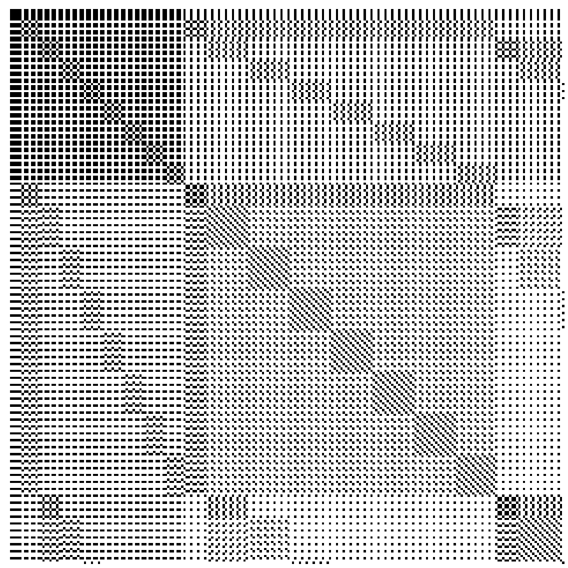
$(a+b+c+d+e+f+g+h)^2$



$(a+b+c+d+e+f+g+h+i)^2$



$(a+b+c+d+e+f+g+h)^3$



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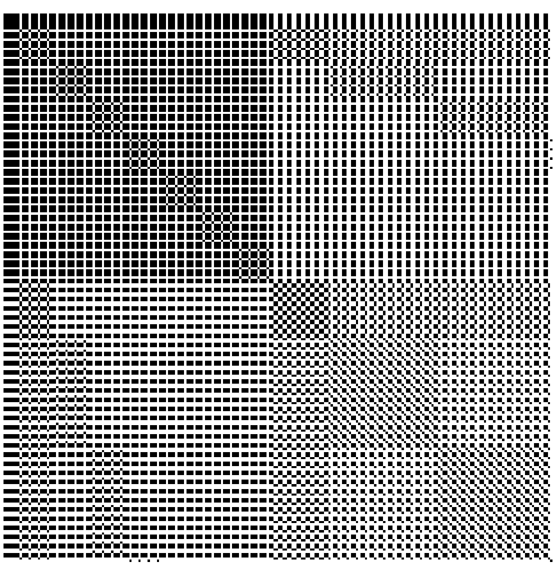
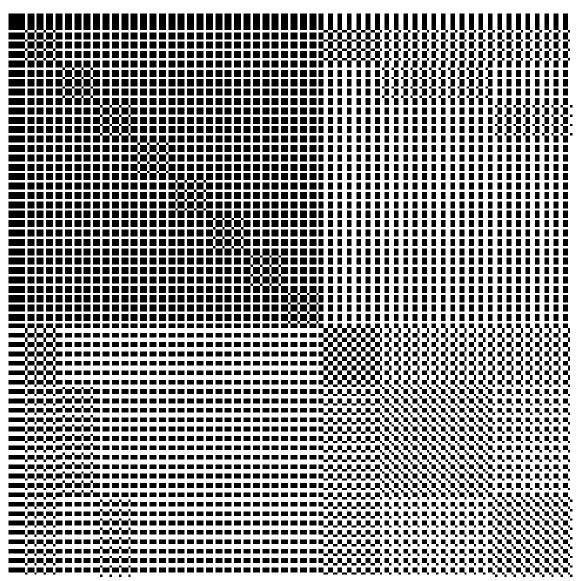


$(a+b+c+d+e+f+g+h)^4$



$(a+b+c+d+e+f+g+h+i)^4$