

Residue Sequences in Weave Design

I don't like nonrepeating decimals. Pi makes me furious.

— Don DeLillo, *Ratner's Star*

Introduction

Most integer sequences that come to mind — such as the integers, the squares, the cubes, the Fibonacci numbers, and the primes — have terms that get larger and larger.

For most design purposes, values need to be limited to a fixed range, such as the number of shafts or treadles available.

The most natural way to bring an integer sequence into a fixed range is to use modular arithmetic — to use the remainders or *residues** of the terms on division by a specified modulus.

For example, the sequence of the cubes

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, 2197, 2744, 3375, 4096, 4913, 5832, 6859, ...

has the following residue sequence mod 10:

1, 8, 7, 4, 5, 6, 3, 2, 9, 0, 1, 8, 7, 4, 5, 6, 3, 2, ...

Periodicity

In addition to bringing sequences into fixed ranges, residues often reveal underlying patterns. The most distinctive pattern is periodicity, in which a fixed number of terms repeats indefinitely. For example, the residue sequence for the cubes mod 10 has period 10 with the repeat:

1, 8, 7, 4, 5, 6, 3, 2, 9, 0

(The fact that the modulus and the period are the same in this case is not a coincidence, but this relationship does not hold in general.)

Some residue sequences have pre-periodic parts before the period begins. An example is the sequence of digits in the decimal expansion of

1/77760

*Strictly speaking, remainders and residues are different things. Their values are the same for nonnegative numbers, but for negative numbers, remainders are negative while residues are expressed in the range 0 to $m - 1$, where m is the modulus. To convert a negative remainder to the corresponding residue, add m to it.

which has the pre-periodic part

0, 0, 0, 0, 1, 2

before settling down to the repeat

8, 6, 0, 0, 8, 2, 3, 0, 4, 5, 2, 6, 7, 4, 8, 9, 7, 1, 1, 9, 3, 4, 1, 5, 6, 3, 7

Periodic sequences without pre-periodic parts are called *purely periodic*.

Although many residue sequences are periodic, some are not. For example, residue sequences for the primes are not periodic for any modulus > 2 . Other residue sequences are periodic only for certain moduli. Furthermore, some residue sequences appear to the eye to be periodic but are not. Examples are the residues of the two Wythoff sequences [1]:

$$a_n = \lfloor n \times \phi \rfloor \quad \text{lower}$$

$$a_n = \lfloor n \times \phi^2 \rfloor \quad \text{upper}$$

where ϕ is the golden ratio, 1.6180339887 ... and $\lfloor x \rfloor$ is the floor of x , the largest integer less than or equal to x .

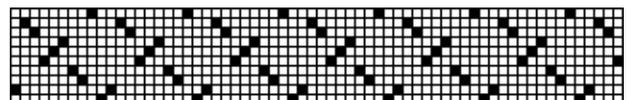
For example, the residue sequence for the lower Wythoff sequence mod 8 is:

1, 3, 4, 6, 0, 1, 3, 4, 6, 0, 1, 3, 5, 6, 0, 1, 3, 5, 6, 0, 1, 3, 5, 6, 0, 2, 3, 5, 6, 0, 2, 3, 5, 7, 0, 2, 3, 5, 7, 0, ...

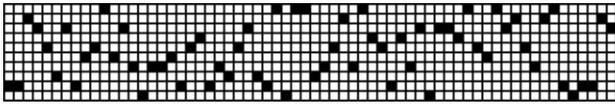
Patterns

The concept of pattern is familiar to everyone, but a precise meaning is elusive and the meaning often depends on context. Nonetheless, our intuitive concept of pattern serves fairly well in practice. We see repetition, as in a periodic sequence, as a pattern. We recognize various symmetries as patterns. And, in general, we perceive order as different from chaos. Of course, not all patterns are attractive.

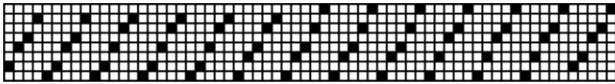
With effort, we can detect patterns in relatively short sequences of integers, but patterns are much easier to detect if the integers are shown by magnitude, as in grid plots. Here are some grid plots of residue sequences in which the bottom row is 0 and the top row is the modulus minus 1.



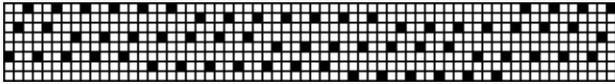
Cubes Mod 10



Fibonacci Numbers Mod 10



Lower Wythoff Sequence Mod 8



Upper Wythoff Sequence Mod 8

Periodic Residue Sequences

Although some non-periodic residue sequences have interesting patterns and hence offer design possibilities, most of the interesting questions involve periodic residue sequences:

- What kinds of sequences yield periodic residue sequences?
- What are their periods for different moduli?
- What residues are present?
- How are the residues distributed?

Although there is no comprehensive answer to the first question, there are some classes of sequences that have periodic residue sequences for all moduli:

- Linear recurrences with constant coefficients, in which each term is expressed as a linear combination of previous terms:

$$a_n = c_1 \times a_{n-1} + c_2 \times a_{n-2} + \dots + c_k \times a_{n-k}$$

The Fibonacci sequence is an example:

$$a_1 = a_2 = 1$$

$$a_n = a_{n-1} + a_{n-2} \quad n > 2$$

- The denominators of continued fraction expansions of quadratic irrationals, such as

$$\sqrt{23} = 4 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{8 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{8 + \frac{1}{1 + \dots}}}}}}}}}$$

whose denominator sequence has the repeat 1, 3, 1, 8.

- The digits of the fractional parts of decimal (and other base) expansions of rational numbers, such as

$$\text{fract}(2/7) = .285714285714285714 \dots$$

which has the repeat 2, 8, 5, 7, 1, 4.

There are, of course, many others, but the ones above have been extensively studied and offer endless possibilities.

Even if a residue sequence is known to be periodic, predicting (as opposed to discovering) its period may be difficult and there are many open questions in this area.

The period often depends on the factors of the modulus. For example, the maximum period for Fibonacci residue sequences is $6 \times m$, where m is the modulus. This maximum is achieved only for moduli of the form

$$m = 2 \times 5^n \quad n > 0$$

That is, $m = 10, 50, 250, \dots$. We know of no formulas for the periods for other forms of m .

A quick glance at a few residue sequences shows that in many cases not all residues are present and that some residues occur more often than others. This has design implications, as we'll show, but as far as we can determine, this matter has not been studied in any detail.

Computing Residue Sequences

Computing residue sequences by hand is tedious and prone to error. One problem with computing residues of a sequence is that the values in many sequences get large very quickly. For example, the Fibonacci sequence, which starts out innocuously as 1, 1, 2, 3, 5, 8, ... quickly gets out of hand — the forty-seventh term is 2,971,215,073, about which we'll say more in a moment.

Most hand-held calculators work with floating point numbers, not integers. Some more sophisticated calculators can perform integer operations, including computing residues, but such calculators have so many other "more important" features that it may be hard, short of buying one, to find out if a particular calculator computes residues. And, while large integers can be done by hand with enough care and patience, calculators have limits on the sizes of integers they can handle.

If you are a programmer, it's easy to write a program to compute residues — most programming languages include a remainder operation in their repertoire of integer arithmetic. However, the size of integers is again a problem, and worse, a program may give the wrong answer for an integer larger than the programming language handles properly. For example, the forty-seventh term in the Fibonacci sequence given above exceeds the word size for a 32-bit computer and some programming languages, including C, don't check for arithmetic overflow. Some programming languages, on the other hand, can handle arbitrarily large integers; if you're conversant with one, computing residue sequences is a snap.

All this aside, if a sequence can be formulated as a recurrence (and many can, even if how to do it is not obvious), large integers can be avoided altogether by not first computing the sequence and then getting the residues, but rather by computing the residue sequence directly.

This is possible because

$$(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$$

$$(a - b) \bmod m = ((a \bmod m) - (b \bmod m)) \bmod m$$

$$(a \times b) \bmod m = ((a \bmod m) \times (b \bmod m)) \bmod m$$

This kind of relationship does not hold, in general, for cancellation of common factors (division). That's an interesting subject but not important here.

What all this means is that residues often can be computed on the fly. For example, Fibonacci residue sequences can be computed by

$$a_n = (a_{n-1} + a_{n-2}) \bmod m$$

Since the residue is taken at each step, intermediate values never exceed $2 \times (m - 1)$.

Residue Sequences in Weave Design

Residues can be used in several ways in weave design. Here we'll stick to their use in designing threading and treadling sequences.

Since residue sequences work on a 0-based system, while shafts and treadles are numbered from 1, it is necessary to convert residue sequences to a 1-based system. All that's needed is to change all 0 values to m , where m is the modulus. To convert 0-based *remainders* to 1-based *residues*, add m to values less than 1 and leave the rest unchanged. Since residues are not negative, the two rules amount to the same thing. See Reference 2 for an explanation.

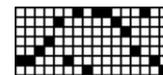
If a residue sequence is periodic, the repeat can be used as a threading or treadling unit. If a residue sequence has a pre-periodic part, that part can be discarded — or used in a variety of ways. If a residue sequence is not periodic, a portion of it can be used.

Some modifications to residue sequences, such as removing duplicate adjacent terms, may be needed to make them suitable for this purpose. Removing duplicates reduces the lengths of repeats, of course.

Where not all residues are present, it may be useful to "fill in" the gaps by moving others into vacant spaces. While this is not necessary if there are enough shafts or treadles without it, it makes clear the resources required and may make patterns easier to see.

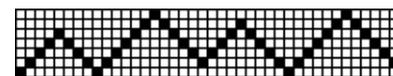
The attached appendix shows the periods of the Fibonacci residue sequences for moduli from 4 through 16, both in their natural form and with duplicates removed and higher values moved down to fill in for missing residues.

Another approach to design using residue sequences is to use the values as "pivots" connected by alternating ascending and descending straight draws. Thus, the Fibonacci sequence mod 7, which has period 16,



6

shifted to a 1-based system with duplicates removed, produces the pivot sequence



7

which has period 40.

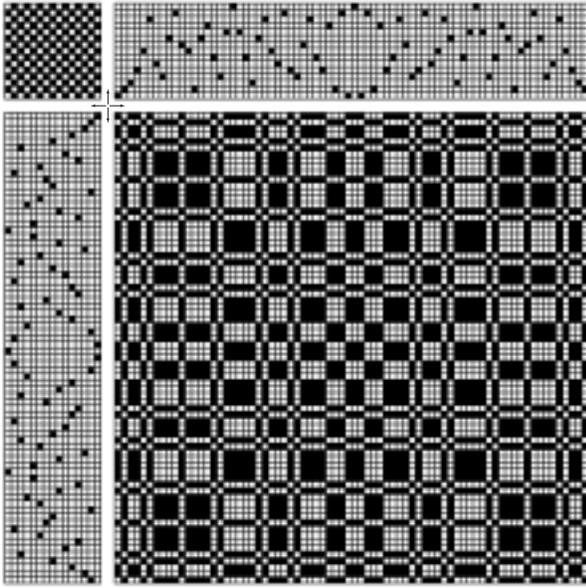
Note that when pivot values increase or decrease in succession, the intermediate values are passed over — only alternating high and low values count.

Examples

A Fibonacci Threading

For the first example, we've used a repeat of the Fibonacci residue sequence for modulus 15, reflected to create a palindrome, as a threading unit. The weave is treadled as drawn in using a tabby tie-up.

The repeat is 74. Here's the draft:

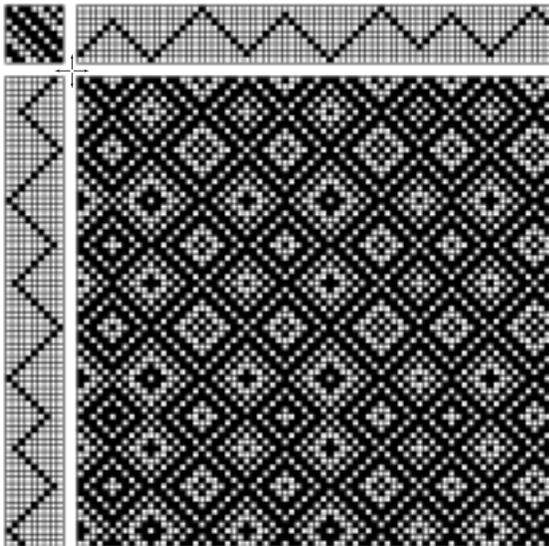


Comment: For drafts based on sequences, we prefer this orientation; the threading reads from left to right and the treadling from top to bottom.

A Wythoff Point Twill

For a second example, we've used 37 terms of the upper Wythoff sequence mod 9 to provide pivots for the threading and then reflected it to form a palindrome. Again, the treadling is as drawn in, with a /2/1/1/2/2/1 twill tie-up.

The repeat is 264. Here's part of the draft:



A reduced drawdown repeat for the entire draft is shown on the last page of this article.

Areas to Explore

There are so many areas to explore that it's hard to even know how to list them – so we'll just mention a few things.

There is, of course, no end to interesting residue sequences. In addition, there are many ways to combine and transform residue sequences to produce new ones.

More interesting, perhaps, are the various ways residue sequences can be used in weave design. So far, we've only mentioned their direct use for threading and treadling sequences and a derivative use in pivot draws. Other possibilities are:

- other derivative uses for designing threading and treadling sequences
- various uses related to profile drafting [3, 4]
- binary (mod 2) residue sequences for designing tie-ups
- binary residue sequences for designing draw-downs
- color selection

What else?

References

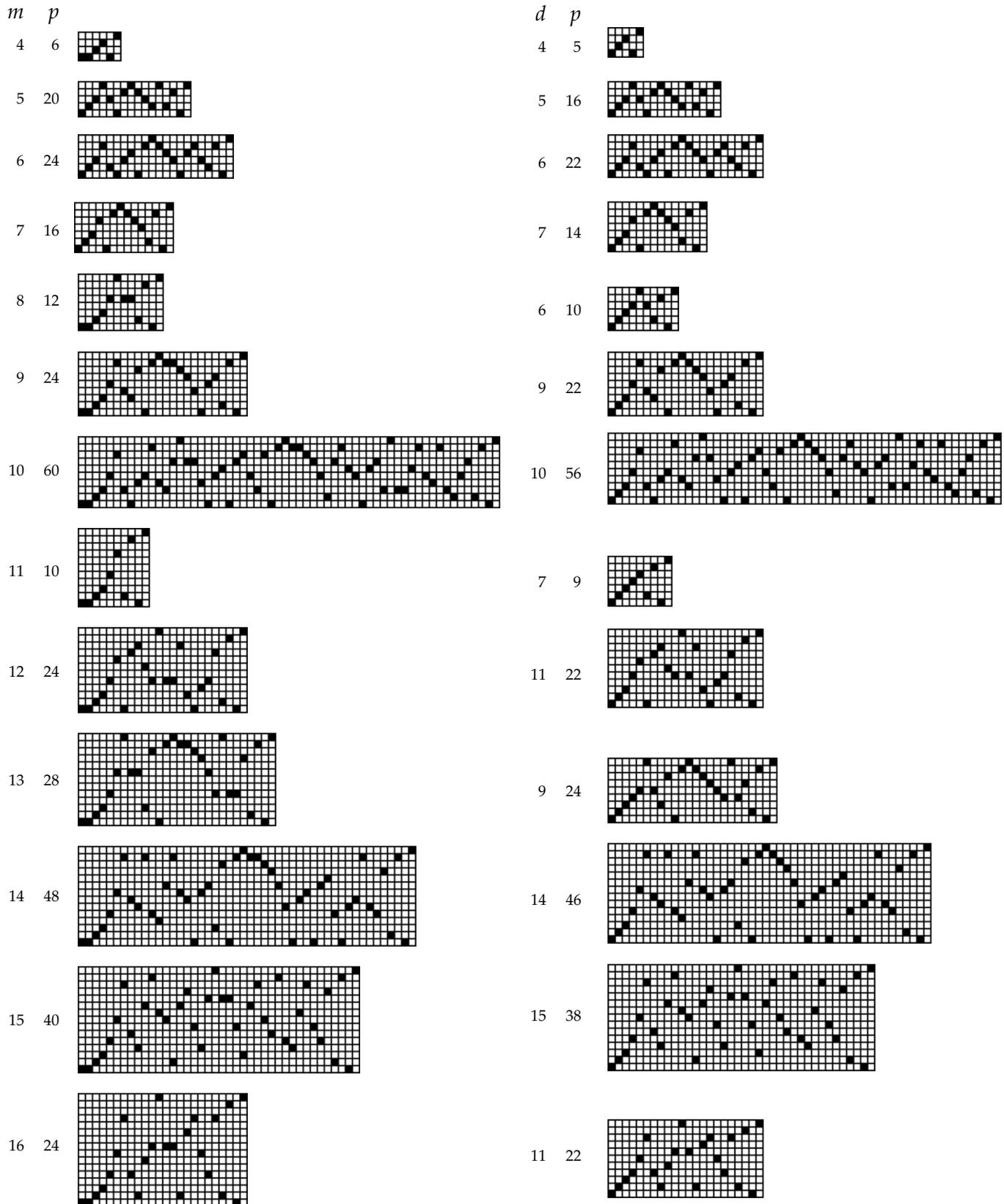
1. *CRC Concise Encyclopedia of Mathematics*, Eric W. Weisstein, Chapman & Hall/CRC, 1998, p. 1952.
2. *Drafting with Sequences*, Ralph E. Griswold, 2004:
http://www.cs.arizona.edu/patterns/weaving/webdocs/reg_seqd.pdf
3. "Block Design the Easy Way", Doramay Keasbey, *Weaver's*, Spring 1997, pp. 6-9.
4. "Beyond Fibonacci: Some Thoughts on Mathematical Sequences and Weaving Drafts", *Complex Weavers Newsletter* 55, September 1997.

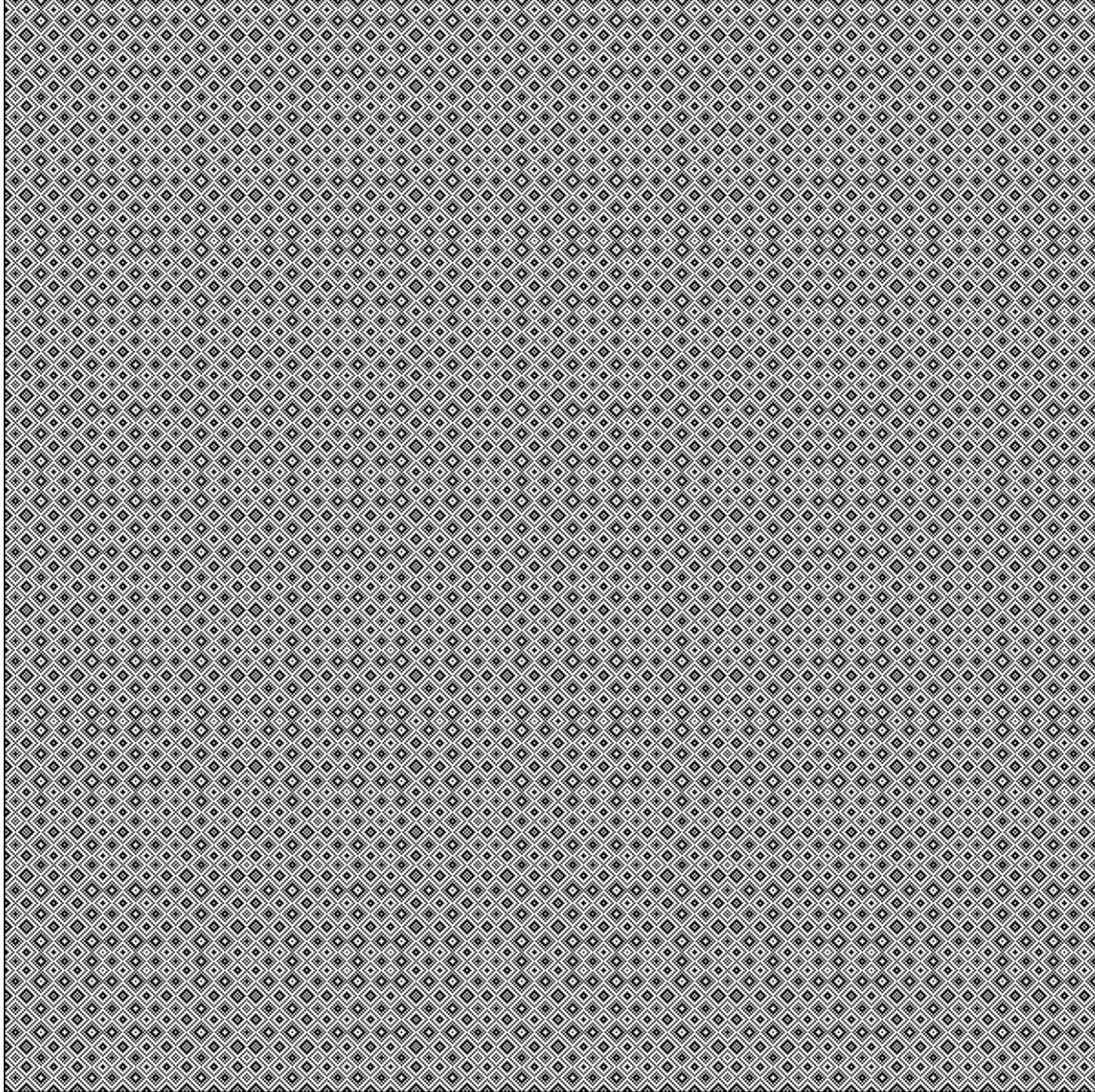
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Appendix — Fibonacci Residue Repeats

All Fibonacci residue sequences are purely periodic. The grid plots in the left column show the repeats. The plots in the right column are the result of removing adjacent duplicate values and moving high values down to fill in the gaps for missing residues. m is the modulus, d is the number of different residues, and p is the period.





Reduced Drawdown Repeat for the Wythoff Point Twill