Declaring Infix Functions...

- The associativity of an operator describes how it binds when combined with operators of equal precedence. So, is
  \[ 5-3+9 \equiv (5-3)+9 = 11 \]
  OR
  \[ 5-3+9 \equiv 5-(3+9) = -7 \]
  The answer is that + and - associate to the left, i.e. parentheses are inserted from the left.
- Some operators are right associative: \( 5^3^2 = 5^{(3^2)} \)
- Some operators have free (or no) associativity. Combining operators with free associativity is an error:
  \[ 5 == 4 < 3 \Rightarrow \text{ERROR} \]

The syntax for declaring operators:

- `infixr prec oper` -- right assoc.
- `infixl prec oper` -- left assoc.
- `infix prec oper` -- free assoc.

```
infixl 7 *
infixl 7 /, 'div', 'rem', 'mod'
infix 4 ==, /=, <, <=, >, >=
```

- An infix function can be used in a prefix function application, by including it in parenthesis. Example:
  \[ ? (+) 5 ((*) 6 4) \]
  \[ 29 \]

- Sometimes it is more natural to use an infix notation for a function application, rather than the normal prefix one:
  - \( 5 + 6 \) (infix)
  - \( (+) 5 ~ 6 \) (prefix)
- Gofer predeclares some infix operators in the standard prelude, such as those for arithmetic.
- For each operator we need to specify its precedence and associativity. The higher precedence of an operator, the stronger it binds (attracts) its arguments: hence:
  \[ 3 + 5*4 \equiv 3 + (5*4) \]
  \[ 3 + 5*4 \not\equiv (3 + 5) * 4 \]
Multi-Argument Functions

- Gofer only supports one-argument functions.
- An $n$-argument function $f(a_1, \cdots, a_n)$ is constructed in either of two ways:
  1. By making the one input argument to $f$ a tuple holding the $n$ arguments.
  2. By letting $f$ “consume” one argument at a time. This is called currying.

Currying

- Currying is the preferred way of constructing multi-argument functions.
- The main advantage of currying is that it allows us to define specialized versions of an existing function.
- A function is specialized by supplying values for one or more (but not all) of its arguments.
- Let’s look at Gofer’s plus operator (+). It has the type $\mathit{(+)} :: \mathit{Int} \rightarrow \mathit{Int} \rightarrow \mathit{Int}$.
- If we give two arguments to (+) it will return an Int:
  $\mathit{(+)} \; 5 \; 3 \Rightarrow 8$
Currying Example...

- \((f'\ 4)\) \(\equiv (f\ 3)\ 4\) returns a function \(f''\ z\) \((f'')\) is a specialization of \(f'\) that adds \(3+4\) to its argument.

\[ f\ 3\ 4\ 5 \equiv ((f\ 3)\ 4)\ 5 \Rightarrow (f'\ 4)\ 5 \Rightarrow f''\ 5 \]

\(f'':\ \text{Int} \to \text{Int} \)
\(f''\ z = 3 + 4 + z\)

- Finally, we can apply \(f''\) to the last argument \(5\) and get the result:

\[ f\ 3\ 4\ 5 \equiv ((f\ 3)\ 4)\ 5 \Rightarrow (f'\ 4)\ 5 \Rightarrow f''\ 5 \Rightarrow 3+4+5 \Rightarrow 12 \]

Currying Example

- Let’s see what happens when we evaluate \(f\ 3\ 4\ 5\), where \(f\) is a 3-argument function that returns the sum of its arguments.

\[ f:\ \text{Int} \to (\text{Int} \to (\text{Int} \to \text{Int})) \]
\[ f\ x\ y\ z = x + y + z \]

\[ f\ 3\ 4\ 5 \equiv ((f\ 3)\ 4)\ 5 \]

Currying Example... The Combinatorial Function:

- The combinatorial function \(\binom{n}{r}\) “n choose r”, computes the number of ways to pick \(r\) objects from \(n\).

\[ \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!} \]

In Gofer:

\[ \text{comb} :: \text{Int} \to \text{Int} \to \text{Int} \]
\[ \text{comb}\ n\ r = \text{fact}\ n/\text{fact}\ r/\text{fact}(n-r) \]

? \text{comb} 5\ 3
10
What’s the Type, Mr. Wolf?

- If the type of a function $f$ is $t_1 \to t_2 \to \cdots \to t_n \to t$
- and $f$ is applied to arguments $e_1 :: t_1, e_2 :: t_2, \cdots, e_k :: t_k$,
- then the result type is given by cancelling the types $t_1 \cdots t_k$:
  - $\vdash f_1 \to f_2 \to \cdots \to f_k \to t_{k+1} \to \cdots \to t_n \to t$
- Hence, $f e_1 e_2 \cdots e_k$ returns an object of type $t_{k+1} \to \cdots \to t_n \to t$.
- This is called the Rule of Cancellation.

Polymorphic Functions

- In Pascal we can’t write a generic sort routine, i.e. one that can sort arrays of integers as well as arrays of reals:

  ```pascal
  procedure Sort (  
    var A : array of <type>;  
    n : integer);  
  end;  
  ```

- In Gofer (and many other FP languages) we can write polymorphic ("many shapes") functions.
- Functions of polymorphic type are defined by using type variables in the signature:

  ```haskell
  length :: [a] -> Int
  length s = ...  
  ```

Associativity

- Function application is left-associative:

  ```haskell
  f a b = (f a) b | f a b \neq f (a b)
  ```

- The function space symbol ‘->’ is right-associative:

  ```haskell
  a -> b -> c = a -> (b -> c)
  a -> b -> c \neq (a -> b) -> c
  ```

- $f$ takes an Int as argument and returns a function of type `Int -> Int`. $g$ takes a function of type `Int -> Int` as argument and returns an Int:

  ```haskell
  f :: Int -> Int -> Int \leftrightarrow f' :: Int -> (Int -> Int)  
g :: (Int -> Int) -> Int  
  ```
Polymorphic Functions...

- Note that `head` and `tail` always take a list as their argument. `tail` always returns a list, but `head` can return any type of object, including a list.
- Note that it is because of Gofer's strong typing that we can only create lists of the same type of element. If we tried to do
  
  \[ ? \ 5 : \ [\text{True}] \]

  the Gofer type checker would complain that we were consing an Int onto a list of Bools, while the type of 
  \[ (:) \ : \ a \rightarrow \ [a] \rightarrow \ [a] \]

Summary

- We want to define functions that are as reusable as possible.
  1. Polymorphic functions are reusable because they can be applied to arguments of different types.
  2. Curried functions are reusable because they can be specialized; i.e. from a curried function \( f \) we can create a new function \( f' \) simply by "plugging in" values for some of the arguments, and leaving others undefined.

Polymorphic Functions...

- `length` is a function from lists of elements of some (unspecified) type \( a \), to integer. I.e. it doesn’t matter if we’re taking the length of a list of integers or a list of reals or strings, the algorithm is the same.

  \[
  \begin{align*}
  \text{length} \ [1,2,3] & \Rightarrow 3 \text{ (list of Int)} \\
  \text{length} \ ["Hi", "there", "]!"] & \Rightarrow 3 \text{ (list of String)} \\
  \text{length} \ "Hi!" & \Rightarrow 3 \text{ (list of Char)}
  \end{align*}
  \]

Polymorphic Functions...

- We have already used a number of polymorphic functions that are defined in the standard prelude.
  
  - `head` is a function from "lists-of-things" to "things":
    \[
    \text{head} :: \ [a] \rightarrow a
    \]
  - `tail` is a function from lists of elements of some type \( a \), to a list of elements of the same type:
    \[
    \text{tail} :: \ [a] \rightarrow [a]
    \]
  - `cons "(:)"` takes two arguments: an element of some type \( a \) and a list of elements of the same type. It returns a list of elements of type \( a \):
    \[
    (:) :: a \rightarrow [a] \rightarrow [a]
    \]
Homework

- Define a polymorphic function \texttt{copy n x} which returns a list of \texttt{n} copies of \texttt{x}.

Example: \texttt{copy 5 "five"}

\texttt{["five","five","five","five","five"]}

\texttt{copy 5 5}

\texttt{[5,5,5,5,5]}

\texttt{copy 5 (dup 5)}

\texttt{[(5,5),(5,5),(5,5),(5,5),(5,5)]}

Homework

- Let \( f \) be a function from \texttt{Int} to \texttt{Int}, i.e. \( f : \texttt{Int} \rightarrow \texttt{Int} \). Define a function \texttt{total f x} so that \texttt{total f} is the function which at value \texttt{n} gives the total \( f(0) + f(1) + \cdots + f(n) \).

Example: \texttt{total double} \texttt{total pow2}

\texttt{double x = 2*x}

\texttt{pow2 x = x^2}

\texttt{totDub = total double}

\texttt{totPow = total pow2}

\texttt{? totDub 5}

\texttt{30}

\texttt{? totPow 5}

\texttt{55}

Summary...

- A polymorphic function is defined using \texttt{type variables} in the signature. A type variable can represent an \texttt{arbitrary} type.

- All occurrences of a particular type variable appearing in a type signature must represent the same type.

- An identifier will be treated as an operator symbol if it is enclosed in backquotes: \	exttt{"\texttt{\textquotesingle}"}.

- An operator symbol can be treated as an identifier by enclosing it in parenthesis: (+).

Homework

- Define a polymorphic function \texttt{dup x} which returns a tuple with the argument duplicated.

Example: \texttt{dup 1}

\texttt{(1,1)}

\texttt{? dup "Hello, me again!"

("Hello, me again!",

"Hello, me again!")}

\texttt{? dup (dup 3.14)}

\texttt{((3.14,3.14), (3.14,3.14))}
Homework

- Define an operator $$ so that $x $$ xs$ returns True if $x$ is an element in $xs$, and False otherwise.

Example:

? 4 $$ [1,2,5,6,4,7]
True

? 4 $$ [1,2,3,5]
False

? 4 $$ []
False