Defining Functions

When programming in a functional language we have basically two techniques to choose from when defining a new function:
1. Recursion
2. Composition

Recursion is often used for basic “low-level” functions, such that might be defined in a function library.

Composition (which we will cover later) is used to combine such basic functions into more powerful ones.

Recursion is closely related to proof by induction.

Defining Functions...

Here’s the ubiquitous factorial function:

```haskell
fact :: Int -> Int
fact n = if n == 0 then 1
else n * fact (n-1)
```

The syntax of a type signature is

```haskell
fun_name :: argument_types
```

`fact` takes one integer input argument and returns one integer result.

The syntax of function declarations:

```haskell
fun_name param_names = fun_body
```

`if e1 then e2 else e3` is a conditional expression that returns the value of `e2` if `e1` evaluates to `True`. If `e1` evaluates to `False`, then the value of `e3` is returned.

Examples:

- `if False then 5 else 6` \( \Rightarrow 6 \)
- `if 1==2 then 5 else 6` \( \Rightarrow 6 \)
- `5 + if 1==1 then 3 else 2` \( \Rightarrow 8 \)
Defining Functions...

- \texttt{fact} is defined recursively, i.e. the function body contains an application of the function itself.
- The syntax of function application is: \texttt{fun_name arg.} This syntax is known as "juxtaposition".
- We will discuss multi-argument functions later. For now, this is what a multi-argument function application ("call") looks like:

  \[ \text{fun\_name arg\_1 arg\_2 \ldots arg\_n} \]

- Function application examples:

  \begin{align*}
  \text{fact}\ 1 & \Rightarrow 1 \\
  \text{fact}\ 5 & \Rightarrow 120 \\
  \text{fact}\ (3+2) & \Rightarrow 120
  \end{align*}

Standard Recursive Functions

- Typically, a recursive function definition consists of a guard (a boolean expression), a base case (evaluated when the guard is \texttt{True}), and a general case (evaluated when the guard is \texttt{False}).

\[
\begin{align*}
\text{fact}\ n &= \\
&\text{if } n == 0 \text{ then } 1 \quad \Rightarrow \text{guard} \\
&\quad \text{else} \\
&\quad n * \text{fact}\ (n-1) \quad \Rightarrow \text{general case}
\end{align*}
\]

Simulating Recursive Functions

- We can visualize the evaluation of \texttt{fact}\ 3 using a tree view, box view, or reduction view.
- The tree and box views emphasize the flow-of-control from one level of recursion to the next.
- The reduction view emphasizes the substitution steps that the \texttt{hugs} interpreter goes through when evaluating a function. In our notation boxed subexpressions are substituted or evaluated in the next reduction.
- Note that the Haskell interpreter may not go through exactly the same steps as shown in our simulations. More about this later.

Tree View of \texttt{fact\ 3}

- This is a Tree View of \texttt{fact\ 3}.
- We keep going deeper into the recursion (evaluating the general case) until the guard is evaluated to True.
Tree View of \texttt{fact 3}

When the guard is True we evaluate the base case and return back up through the layers of recursion.

Box View of \texttt{fact 3}

When the guard is True we evaluate the base case and return back up through the layers of recursion.
Reduction View of \textit{fact} 3

\begin{align*}
\text{fact} 3 & \Rightarrow \\
\text{if } 3 == 0 & \text{ then } 1 \text{ else } 3 \ast \text{ fact } (3-1) \Rightarrow \\
\text{if } \text{False} & \text{ then } 1 \text{ else } 3 \ast \text{ fact } (3-1) \Rightarrow \\
3 \ast & \text{ fact } (3-1) \Rightarrow \\
3 \ast & \text{ fact } 2 \Rightarrow \\
3 \ast & \text{ if } 2 == 0 \text{ then } 1 \text{ else } 2 \ast \text{ fact } (2-1) \Rightarrow \\
3 \ast & \text{ if } \text{False} \text{ then } 1 \text{ else } 2 \ast \text{ fact } (2-1) \Rightarrow \\
3 \ast & (2 \ast \text{ fact } (2-1)) \Rightarrow \\
3 \ast & (2 \ast \text{ fact } 1) \Rightarrow \\
3 \ast & (2 \ast \text{ if } 1 == 0 \text{ then } 1 \text{ else } 1 \ast \text{ fact } (1-1)) \Rightarrow \cdots
\end{align*}

Recursion Over Lists

- In the \textit{fact} function the guard was \texttt{n==0}, and the recursive step was \texttt{fact(n-1)}. I.e. we subtracted 1 from \textit{fact}’s argument to make a simpler (smaller) recursive case.
- We can do something similar to recurse over a list:
  1. The guard will often be \texttt{n==[]} (other tests are of course possible).
  2. To get a smaller list to recurse over, we often split the list into its head and tail, \texttt{head:tail}.
  3. The recursive function application will often be on the tail, \texttt{f tail}.

The length Function

In English:
The length of the empty list \texttt{[]} is zero. The length of a non-empty list \texttt{S} is one plus the length of the tail of \texttt{S}.

In Haskell:
\begin{verbatim}
len :: [Int] -> Int
len s = if s == [] then 0
          else 1 + len (tail s)
\end{verbatim}

- We first check if we’ve reached the end of the list \texttt{s==[]}. Otherwise we compute the length of the tail of \texttt{s}, and add one to get the length of \texttt{s} itself.
Reduction View of $\text{len} \ [5, 6]$

\[
\text{len } s = \text{if } s == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ (\text{tail } s)
\]

\[
\text{len } [5, 6] \Rightarrow
\text{if } [5, 6] == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ ([5, 6]) \Rightarrow
1 + \text{len} \ (\text{tail } [5, 6]) \Rightarrow
1 + \text{len} \ [6] \Rightarrow
1 + (\text{if } [6] == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ ([6])) \Rightarrow
1 + (1 + \text{len} \ [6]) \Rightarrow
1 + (1 + \text{len} \ [\ ])) \Rightarrow
1 + (1 + (\text{if } \emptyset == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ ([\ ]))) \Rightarrow
1 + (1 + 0) \Rightarrow 1 + 1 \Rightarrow 2
\]

Tree View of $\text{len} \ [5, 6, 7]$

\[
\text{len } s = \text{if } s == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ (\text{tail } s)
\]

\[
\text{len } [5, 6, 7] \Rightarrow
\text{if } [5, 6, 7] == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ ([5, 6, 7]) \Rightarrow
1 + \text{len} \ ([6, 7]) \Rightarrow
1 + \text{len} \ [6, 7] \Rightarrow
1 + (\text{if } [6, 7] == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ ([6, 7])) \Rightarrow
1 + (1 + \text{len} \ [6, 7]) \Rightarrow
1 + (1 + (\text{if } [6, 7] == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ ([6, 7]))) \Rightarrow
1 + (1 + 0) \Rightarrow 1 + 1 \Rightarrow 2
\]

Tree View of $\text{len} \ [5, 6, 7]$

\[
\text{len } s = \text{if } s == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ (\text{tail } s)
\]

\[
\text{len } [5, 6, 7] \Rightarrow
\text{if } [5, 6, 7] == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ ([5, 6, 7]) \Rightarrow
1 + \text{len} \ ([6, 7]) \Rightarrow
1 + \text{len} \ [6, 7] \Rightarrow
1 + (\text{if } [6, 7] == \emptyset \text{ then } 0 \text{ else } 1 + \text{len} \ ([6, 7])) \Rightarrow
1 + (1 + \text{len} \ [6, 7]) \Rightarrow
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1 + (1 + 0) \Rightarrow 1 + 1 \Rightarrow 2
\]