Computing Homotopic Shortest Paths Efficiently

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Abstract. We give algorithms to find shortest paths homotopic to given disjoint paths that wind amongst n point obstacles in the plane. Our deterministic algorithm runs in time $O(k \log n + n\sqrt{n})$, and the randomized version in time $O(k \log n + n(\log n)^{1+\varepsilon})$, where k is the input plus output sizes of the paths.

1 Introduction

Geometric shortest paths are a major topic in computational geometry; see the survey paper by Mitchell [15]. A shortest path between two points in a simple polygon can be found in linear time using the "funnel" algorithm of Chazelle [3] and Lee and Preparata [13]. A more general problem is to find a shortest path between two points in a polygonal domain. In this case the "rubber band" solution is not unique, or, to put it another way, different paths may have different homotopy types. When the homotopy type of the solution is not specified, there are two main approaches, the visibility graph approach, and the continuous Dijkstra (or shortest path map) approach [15]. In this paper, we address the problem of finding a shortest path when the homotopy type is specified. Colloquially, we have a "sketch" of how the path should wind its way among the obstacles, and we want to pull the path tight to shorten it.

Homotopic shortest paths are used in VLSI routing [5, 11, 14]. A related problem is that of drawing graphs with "fat edges": given a planar weighted graph G, find a planar drawing such that all the edges are drawn as thickly as possible and proportional to the corresponding edge weights. Duncan *et al.* [7] and Efrat *et al.* [10] present an $O(kn + n^3)$ algorithm for this problem, where n is the number of edges and k is the maximum of their input and output complexities. Hershberger and Snoeyink [12] give an algorithm for the homotopic shortest path problem. Their algorithm assumes a triangulation of size n of the polygonal domain, and finds a shortest path homotopic to a given path of k edges in time linear in k plus the number of triangles (with repetition) visited by the input path. This can be nk in the worst case, and our aim is to reduce it when output size permits. Cabello *et al.* [2] consider the related problem of testing if two given paths are homotopically equivalent. Our original work [9] was done independently of theirs and the idea of the first step of the algorithms is the same. The current presentation of our work incorporates their more efficient implementation of this idea.

We now define homotopy, and give a precise description of our problem. Let $\alpha, \beta : [0, 1] \longrightarrow \mathbb{R}^2$ be two continuous curves parameterized by arc-length. Then

α and β are *homotopic* with respect to a set of obstacles $V \subseteq \mathbb{R}^2$ if α can be continuously deformed into β while avoiding the obstacles; more formally, if there exists a continuous function $h : [0,1] \times [0,1] \to \mathbb{R}^2$ such that:

1. $h(0,t) = \alpha(t)$ and $h(1,t) = \beta(t)$, for $0 \leq t \leq 1$
2. $h(\lambda,0) = \alpha(0) = \beta(0)$ and $h(\lambda,1) = \alpha(1) = \beta(1)$ for $0 \leq \lambda \leq 1$
3. $h(\lambda,t) \notin V$ for $0 \leq \lambda \leq 1, 0 < t < 1$

Let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$ be a set of disjoint, simple polygonal paths and let the endpoints of the paths in Π define the set T of at most $2n$ fixed points in the plane. Note that we allow a path to degenerate to a single point. We call the fixed points of T "terminals," and call the interior vertices of the paths "bends," and use "points" in a more generic sense, e.g. "a point on a path." We assume that no two terminals/bends lie on the same vertical line. Our goal is to replace each path $\pi_i \in \Pi$ by a shortest path σ_i that is homotopic to π_i with respect to the set of obstacles T; see Fig. 1. Note that σ_i is unique. Let $\Sigma = \{\sigma_1, \ldots \sigma_n\}$ be the set of resulting paths. Observe that these output paths may [self] intersect by way of segments lying on top of each other, but will be *non-crossing*.

In order to distinguish how paths go past terminals, we find it convenient to regard each terminal as a small diamond. This actually changes the model of homotopy, because the endpoint of the path is now fixed at one corner of the diamond, and cannot swivel around the terminal. In other words, a path that begins at terminal t and spirals around t before heading off is, with diamond terminals, no longer homotopic to the path that simply heads off. This is the only difference, however, and we can easily remove such spiraling from the final output. Cabello *et al.* [2] call the above two models the "pin" and "pushpin" models, respectively. They also consider the "tack" model, where the endpoints of the paths are not regarded as obstacles.

Let k_{in} be the number of edges in all the paths of Π. Let k_{out} be the number of edges in all the paths of Σ. Note that k_{in} and k_{out} can be arbitrarily large compared to n, and that $k_{out} \leq nk_{in}$. The algorithm of Hershberger and Snoeyink [12] finds homotopic shortest paths in time $O(nk_{in})$. The deterministic algorithm presented in this paper runs in time $O(k_{out} + k_{in} \log n + n\sqrt{n})$, and the randomized algorithm in time $O(k_{out} + k_{in} \log n + n(\log n)^{1+\varepsilon})$. These are improvements except when k_{in} is quite small compared to n.

Our algorithm relies on the algorithm of Bar-Yehuda and Chazelle [1], which uses linear time polygon triangulation and ideas of linear time Jordan sorting, and finds a trapezoidization of n disjoint polygonal chains with a total of k edges in time $O(k + n(\log n)^{1+\varepsilon})$. Replacing this by plane sweep makes our algorithm implementable and yields a running time of $O(k_{out} + (n + k_{in}) \log(n + k_{in}))$ with an extra $O(n\sqrt{n})$ factor for the deterministic version. The algorithm of Cabello *et al.* [2] tests whether two paths are homotopically equivalent under the pin, pushpin, or tack models in $O((n + k_{in}) \log(n + k_{in}))$ time as they opt for the practical plane sweep approach.

In the remainder of this section we give an outline of our algorithm. Although k_{in} can be arbitrarily large compared to n, one easily forms the intuition that,

because the paths are simple and disjoint, k_{in} can be large in a non-trivial way only because path sections are repeated. For example, a path may spiral arbitrarily many times around a set of points, but each wrap around the set is the same. Our method makes essential use of this observation. It seems difficult to search explicitly for repeated path sections. Instead we begin in section 2 by applying vertical shortcuts to the paths (homotopically) so that each left and each right local extreme point occurs at a terminal. These terminals must then be part of the final shortest paths, and we have decomposed the paths into "essential" x-monotone pieces with endpoints at terminals. In section 3 we argue that the number of homotopically distinct pieces is at most $O(n)$. We bundle together all the homotopically equivalent pieces. Routing one representative from each such bundle using the straightforward "funnel" technique takes $O(k_{in}+n^2)$ time total. In section 3 we reduce this using a "shielding technique" where we again exploit the fact that the paths are disjoint and use the knowledge gained in routing one shortest path to avoid repeating work when we route subsequent paths. The final step of the algorithm is to unbundle, and recover the final paths by putting together the appropriate pieces. This is straightforward, and we say no more about it. We summarize the main steps of the algorithm below.The remainder of this paper consists of one section for each of steps 1, 2, and 3 (step i in Section $i+1$).

Summary of the Main Algorithm

1. shortcut paths to divide into essential monotone pieces
2. bundle homotopically equivalent pieces
3. find the shortest path for each bundle
4. unbundle to recover final paths

2 Shortcutting to Find Essential Monotone Pieces

We begin by applying vertical shortcuts to reduce each path to a sequence of essential x-monotone path sections; see Fig. 1. A *vertical shortcut* is a vertical line segment ab joining a point a on some path π with a point b also on π, and with the property that the subpath of π joining a and b, π_{ab}, is homotopic to the line segment ab. We only do *elementary vertical shortcuts* where the subpath π_{ab} consists of [portions of] 2 line segments or 3 line segments with the middle one vertical, and the other two non-vertical. We distinguish *left [right] shortcuts* which are elementary vertical shortcuts where π_{ab} contains a point to the left [right] of the line through ab, and *collinear shortcuts* where π_{ab} lies in the line through ab; see Fig. 2. Collinear shortcuts are applied after left/right shortcuts to eliminate consecutive vertical segments. We in fact only apply *maximal* elementary vertical shortcuts, where the subpath π_{ab} cannot be increased. This means that for left and right shortcuts, either a or b is a bend, or the line segment ab hits a terminal.

A *local left [right] extreme* of a path is a point or, more generally, a vertical segment, where the x-coordinate of the path reaches a local min [max]. Every

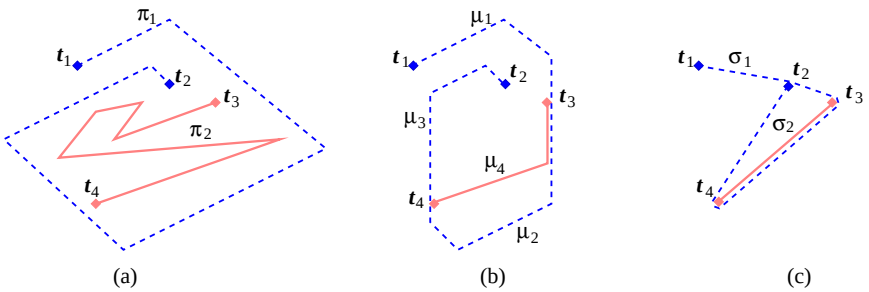

Fig. 1. (a) Paths π_1 and π_2 joining terminals t_1 to t_2 and t_3 to t_4, respectively. (b) After vertical shortcuts, π_1 consists of 3 monotone pieces: μ_1 from t_1 to t_3, μ_2 from t_3 to t_4, and μ_3 from t_4 to t_2; π_2 consists of one x-monotone piece, μ_4, homotopically equivalent to μ_2; (c) Final homotopic shortest paths, σ_1 and σ_2.

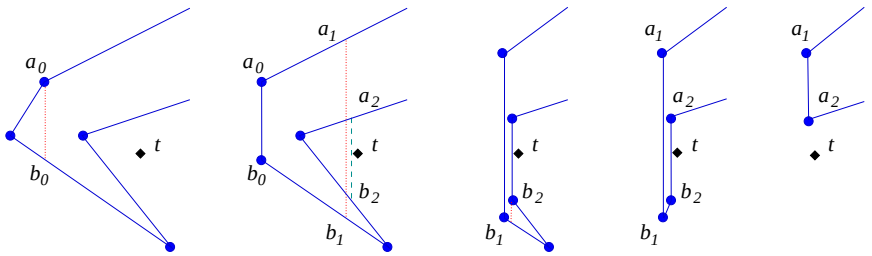

Fig. 2. A sequence of maximal elementary vertical shortcuts (bends are represented by circles and terminals by diamonds). Shortcuts a_0b_0, a_1b_1 and a_2b_2 are left shortcuts; b_2b_1 is a right shortcut; and a_2b_1 and a_1a_2 are collinear shortcuts.

left [right] local extreme provides a left [right] shortcut *unless* the local extreme *is locked at* a terminal, meaning that the left [right] extreme contains the left [right] point of the terminal's diamond. The first two pictures in Fig. 2 show left extremes that provide left shortcuts; the middle picture shows left extremes locked at a terminal.

To detect the maximal left shortcut for a local left extreme, l, we use range searching. Let the segments preceding and following l along the path be l_p and l_f. Let their right endpoints be $r(l_p)$ and $r(l_f)$, and let t be the leftmost of these two points. Then t determines the maximal potential shortcut that can be performed at l in the absence of terminals. We must query the triangle/trapezoid bounded by l, l_p, l_f and the vertical line through t to find the leftmost terminal inside it. This determines the actual maximal left shortcut that can be performed at l. As described in section 2.2, we use an idea from Cabello *et al.* [2] to reduce the range queries to orthogonal range queries, which can be answered more efficiently.

We perform elementary vertical shortcuts until none remain, at which time all local left and right extremes are locked at terminals. We claim that the final

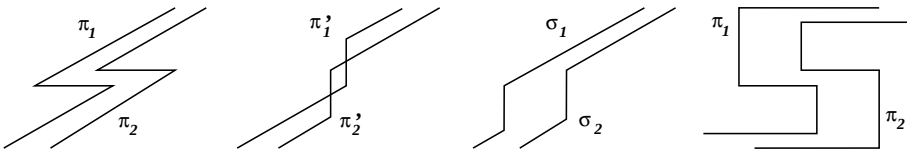

Fig. 3. (a) Disjoint paths π_1 and π_2; (b) Crossing after a left shortcut in π_1 and a right shortcut in π_2. No further shortcuts are possible, hence the crossing cannot be removed; (c) Disjoint paths output by step 1 after two right shortcuts; (d) Rectified version of (a).

shortest paths must be locked in the same way at the same terminals, so we have divided the paths into their *essential monotone pieces*.

Lemma 1 *Let π be a path, and let μ be a result of performing left and right phases of elementary vertical shortcuts on π until no more are possible. Suppose that the local left and right extremes of μ are locked at the terminals $t_{i_1}, \ldots, t_{i_l}$ in that order. Let σ be a shortest path homotopic to π. Then the local left [right] extremes of σ are locked at exactly the same ordered list of terminals, and furthermore, the portion of σ between t_{i_j} and $t_{i_{j+1}}$ is a shortest path homotopic to the portion of μ between those same terminals.*

Proof Sketch. Because π and μ are homotopic, σ is the shortest path homotopic to μ. We can thus go from μ to σ using "rubber band" deformations that only shorten the path, and such deformations cannot loosen a left [right] extreme from the terminal it is locked at. □

Doing elementary vertical shortcuts in an arbitrary order may result in crossing paths; see Fig. 3. To guarantee non-crossing paths we do the elementary vertical shortcuts in alternating *left* and *right phases*. Throughout the algorithm we maintain the sets L and R of local left [right] extremes not locked at terminals. A left phase continues until L is empty. At each step we remove a local left extreme from L, perform the maximal left shortcut there, perform the consequent collinear shortcuts, and update the sets L and R. The new shortcut may destroy one or two elements of R, or create one new element of R or L. These changes occur only at the newly added shortcut and are easy to detect. It is crucial for the correctness of this approach that a left shortcut does not destroy any local left extremes, nor change the shortcuts that will be performed at them. Observe that more than one left and one right phase may be required, since the right phase may add new members to L, for example, in Fig. 2 at least left, right and left phases are needed. Two things remain: to prove that the final paths do not cross, and to analyze the algorithm. We devote a subsection to each.

2.1 Non-crossing paths

Claim 2 *Each left [right] phase of shortcuts preserves the property that paths are non-crossing.*

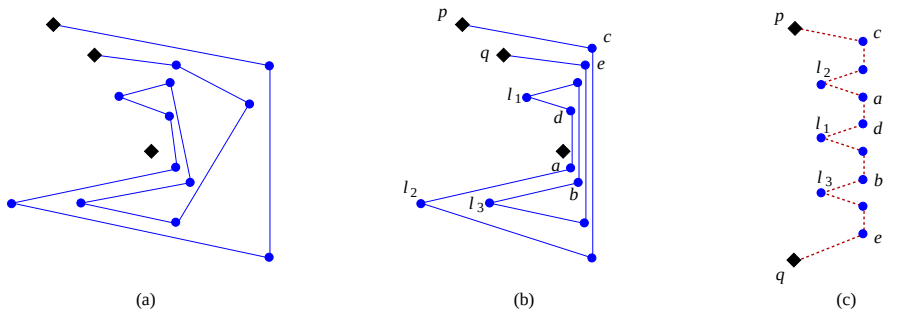

Fig. 4. (a) A path undergoes a right phase; (b) Handling local left extreme l_1 first yields the shortcut ab. Handling the local left extreme l_2 first yields shortcut cd. The final result of handling l_1, l_2 and l_3 in any order is shortcut ce; (c) An "unfolded" version of part (b).

It is crucial for steps 2 and 3 of our algorithm that the paths remain non-crossing. By symmetry, we can concentrate on a left phase. Note that a left phase, as described, is non-deterministic. At each stage we choose one member l from the set L of current local left extremes, and use it to perform a left shortcut. To prove the Lemma we first show that for each left phase there is *some* sequence of choices that preserves the property that the paths are non-crossing. We then argue that the end result of a phase does not depend on the choices made.

Claim 3 *Performing left shortcuts in rightmost order of the local left extremes leaves the paths non-crossing.*

Observe that we could complete a left phase of shortcuts using this "rightmost" order, at a cost of an extra $\log k_{in}$ factor for maintaining a heap. There is no practical advantage to doing this.

Lemma 4 *The end result of a left phase does not depend on the sequence of choices made during the phase.*

Proof. We begin with the claim that the set $\mathcal{L}$ of all local left extremes that appear in L over the course of the phase is independent of the choices made during the phase. This implies that the set of left shortcuts performed during the course of the phase is also independent of the choices made during the phase. However, the set of collinear shortcuts is *not* independent. In particular, the order in which we perform left shortcuts affects the set of collinear shortcuts; see Fig. 4.

Consider one left phase. Let L_0 be the initial set of local left extremes not locked at terminals. Let $\mathcal{L}$ be the union of L over the course of the phase. Suppose two sequences of choices C_1 and C_2 during a left phase yield sets $\mathcal{L}_1$ and $\mathcal{L}_2$. We would like to show that $\mathcal{L}_1 = \mathcal{L}_2$. Consider $l \in \mathcal{L}_1$. We will prove $l \in \mathcal{L}_2$ by induction on the number of left shortcuts performed in the phase before l enters $\mathcal{L}_1$. If this number is 0 then $l \in L_0$ and we are done. Otherwise, l is added to L

as a result of some left shortcut and consequent collinear shortcuts. Any vertical segment used in the collinear shortcuts may, in its turn, have arisen as a result of some left shortcut and consequent collinear shortcuts. Tracing this process, we find that l is formed from a set of left shortcuts linked by vertical segments, all in the same vertical line as l; see Fig. 4. All these left shortcuts arose from local left extremes that entered $\mathcal{L}_1$ before l did, and thus, by induction, are in $\mathcal{L}_2$. They only leave $\mathcal{L}_2$ when we perform their shortcuts, and thus the choice sequence C_2 includes these shortcuts—though possibly in a different order than in C_1. The consequent collinear shortcuts will merge all the verticals forming l, and cannot merge more than that because the segments attached before and after l are not vertical (l is a local left extreme). Thus l is in $\mathcal{L}_2$.

This proves that the set of local left extremes, and thus the set of left shortcuts is independent of choices made during the phase. Any vertical segment that is in the final set of paths output by the phase arises through left shortcuts plus consequent collinear shortcuts. Since any set of choices leads to the same set of left shortcuts, (possibly in different order), the consequent collinear shortcuts will arrive at the same final vertical shortcuts—i.e. the same final paths. □

2.2 Implementation and run-time analysis

We use range queries to identify the maximal shortcut that can be performed at a local extreme of the path. Our original version of this paper [9] used general simplex range searching, which is time-consuming. Cabello *et al.* [2], in their paper on testing homotopy, independently developed this same idea of shortcutting to divide a path into essential monotone pieces. However, they were able identify shortcuts using *orthogonal* range queries, which can be done much more efficiently. The main insight is that shortcuts depend only on the *aboveness* relation among terminals and x-monotone path sections, and that the aboveness relation forms a partial order. This partial order can be preserved while each monotone path section is modeled as a horizontal line segment, and the whole picture becomes "rectified"; see Fig. 3(d). Shortcuts can now be identified using orthogonal range queries.

We find the aboveness relation from a trapezoidization of the input, which can be computed in time $O(k_{itin} + n(\log n)^{1+\varepsilon})$ using the algorithm of Bar-Yehuda and Chazelle [1]. The partial order can be extended to a total order in linear time. We now assign to each terminal and x-monotone path section, a new y-coordinate equal to its rank in the total order. As justified in [2], shortcuts after this transformation correspond to shortcuts in the original. Each shortcut now requires an orthogonal range query: given an axis-aligned query rectangle, certify that it is empty or else return the rightmost [leftmost] terminal inside it. Chazelle's data structure for segment dragging [4] solves this with $O(n \log n)$ preprocessing time, $O(n)$ space, and $O(\log n)$ query time. Simpler algorithms require an extra logarithmic factor in space [8, 16]. We need one more ingredient to bound the running time of step 1 of our algorithm by $O(k_{in} \log n + n(\log n)^{1+\varepsilon})$.

Lemma 5 *The number of elementary vertical shortcuts that can be applied to a set of paths with a total of k_{in} segments is at most $2k_{in}$.*

Proof. Assume that no two terminals and/or bends line up vertically. Consider the set of vertical lines through bends and through the left and right sides of each terminal's diamond. An elementary shortcut operates between two of these vertical lines. If a left [right] shortcut has its leftmost [rightmost] vertical at a bend, then after the shortcut, this vertical disappears forever; see Fig. 2. There are thus at most k_{in} such elementary shortcuts. Consider, on the other hand, a left shortcut that has its leftmost vertical at a terminal. This only occurs when two previous left shortcuts are stopped at the terminal, and then combined in a collinear shortcut. See the right hand pictures of Fig. 2. Thus an original edge of the path has disappeared. Note that elementary shortcuts never fragment an edge of the path into two edges, but only shorten it from one end or the other. Thus there are at most k_{in} elementary shortcuts of this type. Altogether, we obtain a bound of $2k_{in}$ elementary shortcuts. □

3 Bundling Homotopically Identical Paths

Let $M = \{\mu_1 \dots \mu_m\}$ be the set of essential x-monotone paths obtained from step 1 of our algorithm. In the second step of the algorithm we bundle homotopically equivalent paths in M. More precisely, we take one representative path for each equivalence class of homotopically equivalent paths in M. This is justified because the paths in each equivalence class have the same homotopic shortest path. Because the paths in M are non-crossing and x-monotone, it is easier to detect homotopic equivalence: two paths are homotopically equivalent if they have the same endpoints, and, between these endpoints no terminal lies vertically above one path and vertically below the other.

We perform the bundling using a trapezoidization of M; see Fig. 5. We apply the trapezoidization algorithm of Bar-Yehuda and Chazelle [1] to the paths obtained after the shortcuts of step 1, but before these paths are chopped into monotone pieces. We claim that we can perturb the paths output from step 1 so that they become disjoint and the Bar-Yehuda and Chazelle algorithm can be applied. The trapezoidization algorithm takes time $O(k_{in} + n(\log n)^{1+\varepsilon})$. Once we have a trapezoidization of M, we can in linear time bundle homotopically equivalent paths as follows. While scanning each $\mu \in M$, we check if it is homotopically equivalent to the path "below" it, by examining all the edges of the trapezoidization that are incident to μ from below. If all these trapezoidization edges reach the same path μ_j and none pass through a terminal on the way to μ_j, and μ_i and μ_j have the same terminals as endpoints, then we mark μ_i as a duplicate. Let $R = \{\rho_1 \dots \rho_r\}$ be the paths of M that are not marked as duplicates.

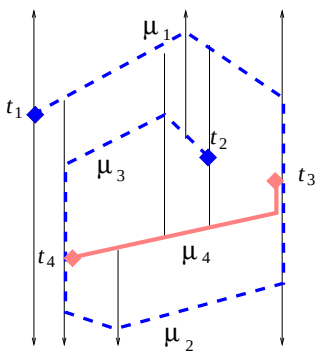

Fig. 5. A trapezoidization helps verify that μ_2 and μ_4 are equivalent.

Lemma 6 *The number of paths in R is bounded by $2n$.*

Proof. Note that every terminal t is either a right endpoint of paths in R or a left endpoint of paths in R, but not both. This is because a terminal cannot have local left extremes and local right extremes locked at it without the paths crossing.

For each terminal $t \in T$, associate its bottommost incident path $\phi(t)$ (its "floor"), and the first path hit by a vertical ray going up from t, $\gamma(t)$ (its "roof"). We claim that every path in R is $\phi(t)$ or $\gamma(t)$ for some $t \in T$. This proves that the number of paths in R is at most $2n$.

Consider a path $\rho \in R$ with left and right terminals s and t, respectively. Suppose that ρ is not a roof. Then every vertical ray extending downward from a point of ρ must hit the same path σ. (If two rays hit different paths, then in the middle some ray must hit a terminal.) Furthermore, since ρ is not the bottommost path incident to s, σ must hit s. Similarly σ must hit t. But then σ and ρ are homotopically equivalent. □

4 Shortest Paths

In this section we find shortest paths homotopic to the monotone paths $R = \{\rho_1, \ldots, \rho_r\}$ produced in the previous section. Note that r is $O(n)$. Let ρ'_i denote the shortest path homotopic to ρ_i. We route each path using a funnel technique. The funnel algorithm of [3] and [13] operates on a triangulation of the n points (the terminals in our case), and follows the path through the triangulation maintaining a current "funnel" containing all possible shortest paths thus far. The algorithm takes time proportional to the number of edges in the path plus the number of intersections between the triangulation edges and the path.

Rather than a triangulation, we will use a trapezoidization formed by passing a vertical line through each of the n terminals; see Fig. 6(a). Then, since each path ρ_i is x-monotone, it has $O(n)$ intersections with trapezoid edges, and the funnel algorithm takes time proportional to n plus the number of edges in ρ_i. Since the total number of edges in R is bounded by k_{in}, this gives a total over all paths of $O(n^2 + k_{in})$. (Note that the number of edges in the output is $O(n^2)$, so it doesn't appear explicitly here.)

We improve this by making use of the fact that the paths are disjoint, and re-using information gained when each path is routed. We give a randomized algorithm to route the paths of R in time $O(n \log n + k)$, and a deterministic algorithm with $O(n\sqrt{n} + k)$ running time. Recall that $k = k_{in} + k_{out}$.

Both methods use a "shielding" technique. We begin by describing this idea intuitively. First note that the ρ_i's can be routed independently, since none affects the others. The initial paths ρ_i are non-crossing, and so are the final shortest paths, ρ'_i. If ρ_j is below ρ_i, and we have already computed ρ'_j, then ρ'_j behaves as a barrier that "shields" ρ_i from terminals that are vertically below ρ'_j; see Fig. 6(b).

To utilize shielding we will modify the basic trapezoidization described above as we discover shortest paths ρ'_i. In particular, the upward vertical ray through

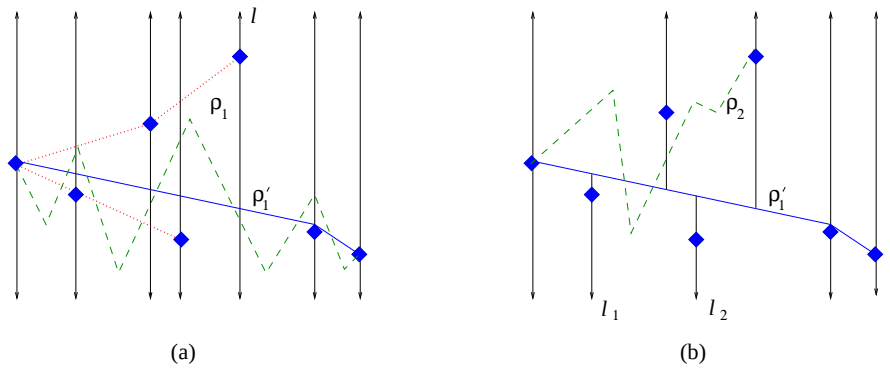

Fig. 6. (a) A path ρ_1 (dashed), its funnel up to line l (dotted) and the final homotopic shortest path ρ_1'; (b) After the shortest path ρ_1' has been found, shielding allows us to route another path ρ_2 without examining vertical lines l_1 and l_2.

terminal t, $u(t)$, will be truncated at the lowest shortest path ρ_i' that is strictly above t and does not bend at t. The downward vertical ray through terminal t, $d(t)$, will be truncated in an analogous way. To route a new path ρ_i through this modified trapezoidization we use the following algorithm.

Routing ρ_i with shielding

1. Identify the first trapezoid that ρ_i' will traverse. This can be done in $O(\log n)$ time because the shortest paths observe the same vertical ordering as the original ρ_j's.
2. Traverse from left to right the sequence of trapezoids that ρ_i' will pass through. (Note that ρ_i itself may pass through different trapezoids; see Fig. 6(b).) We construct the funnel for ρ_i' as we do this traversal. Suppose that our path enters trapezoid τ. To leave τ on the right we have two cases. If the right side of τ is a point, then it is a terminal t, and we are locked between two paths that terminate or bend at t. Then the funnel collapses to this point, and we proceed to the next trapezoid if ρ_i continues. Otherwise the right side of τ is a vertical through some terminal t, and (unless ρ_i ends at t) we have a choice of two trapezoids to enter, the upper one with left side $u(t)$ or the lower one with left side $d(t)$. We follow path ρ_i until it crosses the infinite vertical line through t. If it passes above t then we enter the upper trapezoid, and otherwise we enter the lower trapezoid. We update the funnel to include t.
3. When we reach the right endpoint of ρ_i, the funnel gives us the shortest homotopic path ρ_i'.
4. We update the trapezoidization as follows. For each vertical segment $u(t)$ or $d(t)$ that is intersected by ρ_i' we chop the segment at its intersection point with ρ_i', provided that the intersection point is not t (i.e. that ρ_i' does not bend at or terminate at t).

Without yet discussing the order in which we route the paths, we can say a bit about the timing of this shielding method. As we traverse a path ρ_i we spend time proportional to the size of ρ_i plus the number of trapezoids traversed by ρ_i'. When ρ_i' leaves a trapezoid, say at the line segment $u(t)$ above terminal t, it may happen that ρ_i' will bend at t or terminate at t. In this case t is part of the output path ρ_i', and we can charge the work for this trapezoid to the output size k_{out}. If, on the other hand, ρ_i' does not bend or terminate at t, then it crosses $u(t)$ and we chop $u(t)$ there. In this case we charge the work for this trapezoid to the chop. Thus the total time spent by the algorithm is $O(k + C)$ where C is the total number of chops performed at the n verticals.

For shielding to be effective we need to route paths in an order that makes C grow more slowly than n^2. We first analyze the randomized algorithm where we choose the next path to route at random with uniform probability from the remaining paths.

Lemma 7 *If paths are routed in random order then the expected number of times we chop a segment $u(t)$ or $d(t)$ is $O(\log r)$, and thus C is $O(n \log n)$, and the routing takes time $O(k + n \log n)$.*

Proof. This follows from a standard backward analysis. See for example [6] for other similar proofs. Let l be a vertical line through a point t, and let u denote a ray emerging vertically from t. Assume that m paths intersecting u have been inserted up to now, and we are about to insert a new one. Then u is chopped if and only if the new path creates an intersection point below all existing intersection points on u, but above t itself. Since the order of the insertions is random, the probability of the new intersection point being below all other intersection points is $1/m$. Summing over all paths yields the claimed bound. □

Finally, we describe how to achieve a routing time of $O(k+n\sqrt{n})$ deterministically. Suppose that the paths $\{\rho_1, \ldots, \rho_r\}$ are in order from top to bottom. (We get this for free from step 2 of our algorithm.) Partition the paths into blocks $B_1, \ldots, B_{\sqrt{r}}$ each of size $\sqrt{r}$, and route them one block at a time from $B_{\sqrt{r}}$ to B_1. Within each block we process the paths in order. The largest increasing [decreasing] sequence with this ordering has size $\sqrt{r}$, therefore the number of chops at each vertical is $\sqrt{r}$. Thus C is $O(n\sqrt{n})$, and this routing method takes time $O(k + n\sqrt{n})$.

5 Conclusion and Open Problems

Given n terminal points in the plane, and a set of disjoint paths joining pairs of the terminals, we can find shortest paths homotopic to the originals in time $O(k \log n + n\sqrt{n})$, or with randomization, in time $O(k \log n + n(\log n)^{1+\varepsilon})$, where k is the sum of input and output sizes of the paths. Unless k grows quite slowly compared to n, this is better than the Hershberger-Snoeyink algorithm which runs in time $O(nk)$.

One open problem is to improve the running time of our deterministic algorithm. The bottleneck is the $O(n\sqrt{n})$ arising from the shielding technique in

Section 4. There is no reason to think that $O(n\sqrt{n})$ is the "right" answer—it is just the best one can do by ordering the paths while remaining otherwise oblivious to them.

Another open problem is to extend our efficient shortest homotopic path algorithm to the "tack" model, as described in [2]. Under this model the endpoints of paths need not be obstacles; they act as tacks that are pushed all the way into the backing, allowing the rubber bands of the paths to pass freely over them. Cabello *et al.* [2] successfully use the shortcut technique to find the essential x-monotone pieces so they can test equivalence of two paths under the tack model. However, the problem of finding shortest paths seems more difficult, because the final shortest paths cross in general, and the later steps of our algorithm fail.

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