3D Proportional Contact Representations of Graphs

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***Abstract*—In 3D contact representations, the vertices of a graph are represented by 3D polyhedra and the edges are realized by non-zero-area common boundaries between corresponding polyhedra. While contact representations with cuboids have been studied in the literature, we consider a novel generalization of the problem in which vertices are represented by axis-aligned polyhedra that are union of two cuboids. In particular, we study the weighted (proportional) version of the problem, where the volumes of the polyhedra and the areas of the common boundaries realize prespecified vertex and edge weights. For some classes of graphs (e.g., outerplanar, planar bipartite, planar, complete), we provide algorithms to construct such representations for arbitrary given weights. We also show that not all graphs can be represented in 3D with axis-aligned polyhedra of constant complexity.**

I. Introduction

Realizing graphs as contact graphs of geometric objects has been a subject of study for many decades. In a *contact representation* of a graph G, the vertices of G are usually 2D geometric objects such as discs, segments or curves, while the edges of G are realized by two objects touching in some prescribed fashion. Such representations are preferred in some contexts over the standard node-link representations for displaying relational information, as it provides the viewer with the familiar metaphor of geographical maps [16].

In the 2D vertex-weighted version of the problem the input also specifies a function $w : V \to \mathbb{R}^+$ assigning weights to vertices the input graph $G = (V, E)$. The goal is to construct a contact representation of G so that each vertex is represented by an object whose area realizes the specified weight. In the edge-weighted variant, the lengths of contacts between two objects should be equal to the prescribed edge weights $w : E \to \mathbb{R}^+$. Such value-by-area contact representations are called *proportional*. In this paper, we study a natural generalization of the problem and consider contact representations of graphs in 3D, in which vertices are represented by polyhedra and edges are represented by shared boundary between the corresponding polyhedra. As in the 2D case, we investigate the weighted version of the problem, where in addition to a graph $G = (V, E)$, the input may also contain vertex and edge weights. Depending on the input data, we distinguish three problem variants:

1) in the vertex-weighted scenario, we are given a function $w : V \to \mathbb{R}^+$ and the goal is to represent G so that volume of the polyhedron for a vertex $v \in V$ equals $w(v)$;
2) in the edge-weighted scenario, weights $w : E \to \mathbb{R}^+$ are given, and in the resulting representation, the surface area of the contact between $u \in V$ and $v \in V$ equals $w(u, v)$.
3) in the vertex-edge-weighted scenario, both the volumes of the vertices and the areas of the contacts should be realized for the given weights $w : V \cup E \to \mathbb{R}^+$.

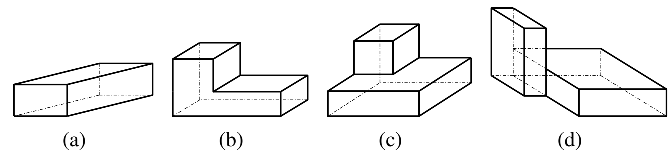

Fig. 1. The simplest axis-aligned 3D polyhedra: (a) a box, (b) a normal $\mathcal{L}$, (c) a box with a leg, (d) a broken $\mathcal{L}$.

TABLE I. Complexity of various 3D axis-aligned objects.

shape	corners	edges	faces
box	8	12	6
normal $\mathcal{L}$	12	18	8
box with a leg	14	21	9
broken $\mathcal{L}$	15	23	10

A contact representation can be a compelling visualizations of the underlying graph; for cognitive and practical reasons, a natural goal is to limit the complexity of the polyhedra used in the representation. For axis-aligned polyhedra, the notion of *polyhedral complexity* can be measured by the number of cuboids that make a polyhedra, or by the number of corners, edges and faces in the polyhedra. Previous results on axis-aligned contact representations in 3D only consider axis-aligned cuboids to represent vertices [6], [15], [22]. However, there are graphs, even as simple as K_5, for which cuboids are not sufficient. We thus address the problem of contact representation with axis-aligned polyhedra consisting of two cuboids. In particular we consider three types of polyhedra, which we call a *normal* $\mathcal{L}$, a *box with a leg*, and a *broken* $\mathcal{L}$; see Fig. 1. It is not hard to see that these are the simplest axis-aligned polyhedra in 3D after cuboids; see Table I for a comparison of the complexity of the different types of polyhedra.

A. Related Work

The study of contact graphs dates back to 1936 when Koebe [19] showed that planar graphs can be realized as disc contact graphs. Since then contact graphs in the plane have been extensively studied for curves [17], segments [10], and polygons [7], [11], [13], [20], [24]. The weighted variant of the problem in the plane has also been extensively studied [1], [5], [14], [21], [23]. On the other hand, there is much less known about contact representations in the 3D space. Contact graphs using 3D objects have been studied for spheres [4], [18], cylinders [3], tetrahedra [2] and axis-aligned cuboids [6], [15], [22]. Thomassen shows that any planar graph has a contact representation by cuboids [22]. Recently, Bremner et al. [6] showed two alternative proofs for this claim using canonical orders and Schnyder realizers. They also studied the unit-cube

TABLE II. Overview of our results: polyhedral complexity (number of corners) of 3D axis-aligned objects for contact representation.

graph class	vertex-weighted	edge-weighted	vertex-edge-weighted
outerplanar	12 (Thm. 1)	12 (Thm. 2)	15 (Lm. 5)
planar bipartite	8 (Thm. 3)	8 (Thm. 3)	8 (Thm. 3)
planar	12 (Thm. 1)		
K_n	15 (Lm. 1)		
$K_{n,m}$	8 (Lm. 2)		

version of the problem and showed that any graph with n vertices, admitting a contact representation by unit cubes, has at most $7n - \Omega(n^{2/3})$ edges, and the bound is tight. Felsner and Francis [15] relax the non-zero-area contact requirement and show that any planar graph has a contact representations with cubes where some of the contacts might be improper, that is, of zero area.

B. Our Contribution

We investigate proportional contact representations of graphs with axis-aligned polyhedra, that are the union of at most two cuboids. To the best of our knowledge, there is no prior work on contact representation with axis-aligned polyhedra more complex than cuboids. Furthermore, none of the previous results address the weighted variant of the problem in 3D. For the vertex-weighted variant, we show that any planar graph admits a representation with normal $\mathcal{L}$'s. There exist representations for any complete graph with broken $\mathcal{L}$'s and for any complete bipartite graph with cuboids. On the negative side, we show that not all subgraphs of complete graphs admit a contact representation with axis-aligned polyhedra of constant complexity. For the edge-weighted variant, we prove that normal $\mathcal{L}$'s are sufficient to represent any outerplanar graph. For the vertex-edge-weighted variant, any planar bipartite graph has a vertex-edge-weighted representation with cuboids. In contrast, a graph containing K_4 as a subgraph may not have such a representation. Finally, any outerplanar graph admits a vertex-edge-weighted contact representation with broken $\mathcal{L}$'s. Our results are summarized in Table II.

II. Vertex-Weighted Representations

First we focus on proportional contact representations of planar graphs.

A. Planar Graphs

Theorem 1: Let $G = (V, E)$ be a planar graph and let $w : V \to \mathbb{R}^+$ be a weight function. Then G has a proportional contact representation with respect to w with normal $\mathcal{L}$'s.

Proof: For a rectangular cuboid B, the *length* (*width, height*, resp.) of B is the x-span (y-span, z-span, resp.) of B. Also define the *left*, *right*, *back*, *front*, *bottom* and *top* of B as the faces with the minimum x-coordinate, maximum x-coordinate, minimum y-coordinate, maximum y-coordinate, minimum z-coordinate and maximum z-coordinate, respectively. Assume that G is a maximal planar graph, since otherwise we can add dummy vertices to make it maximal, and then from a representation of the resulting super graph, delete $\mathcal{L}$'s for the dummy vertices to obtain the desired representation.

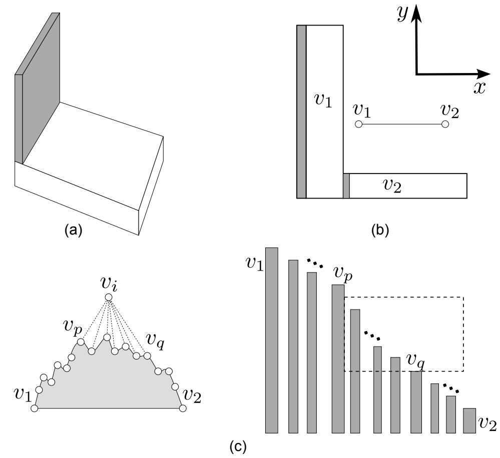

Fig. 2. Proof of Theorem 1. (a) A normal $\mathcal{L}$ with a base and a pillar (shaded). (b) Placing $\mathcal{L}$'s for v_1 and v_2, z-axis is directed towards a viewer. (c) Inserting a new vertex v_i into the staircase profile.

We first construct a contact representation Γ of G with normal $\mathcal{L}$'s. We then modify Γ so that each $\mathcal{L}$ for a vertex v has volume $w(v)$. To construct the contact representation Γ, we use the concept of canonical ordering for the vertices of a planar graph [12]. Recall that a *canonical order* $v_1 = u$, $v_2 = v$, $v_3, \ldots, v_n = w$ for the vertices of a maximal planar graph $G = (V, E)$ with outer vertices u, v and w (in clockwise order) is one that meets the following criteria for every $4 \le i \le n$.

- The subgraph $G_{i-1} \subseteq G$ induced by $v_1, v_2, \ldots, v_{i-1}$ is biconnected, and the boundary of its outer-face is a cycle C_{i-1} containing the edge (u, v).
- The vertex v_i is in the exterior face of G_{i-1}, and its neighbors in G_{i-1} form a (at least 2-element) subinterval of the path $C_{i-1} - (u, v)$.

Let $\Pi = (v_1, v_2, \ldots, v_n)$ be the vertices in a canonical ordering of G, which can be computed in linear time [8]. We construct Γ by inserting $\mathcal{L}$'s for the vertices of G in the order of Π. Each $\mathcal{L}$ in the representation has the same orientation: it has a rectangular *base* of unit height and attached to the top of this base at its left side; there is a *pillar*, which has width exactly equal to the width of the base and very small length; see Fig. 2(a). At each step of the construction, the top of the representation maintains a "staircase profile" in 2D, formed by the tops of the pillars for all the "active" vertices. Here the *active* vertices after i-th step are the vertices on the outer-face of G_i, that is, the vertices which have neighbors in $G - G_i$. The *staircase profile* for these vertices refers to the property that the front side of the pillars for these vertices forms a staircase with incrementally increasing x-coordinate and decreasing y-coordinate, as we go from left to right along the path $C_i - \{(u, v)\}$; see Fig. 2(c).

We start the construction by representing v_1 and v_2 by two normal $\mathcal{L}$'s so that the right side of the base of v_1 touches the left side of the base of v_2; see Fig. 2(b). Note that the tops of the pillars for these two vertices maintain the desired staircase profile. Now consider inserting a new vertex v_i. By the property of canonical order Π, the vertices of G_{i-1} that are neighbors of v_i are all consecutive on $C_{i-1} - \{(1, 2)\}$. Let v_p and v_q be the leftmost and the rightmost among these

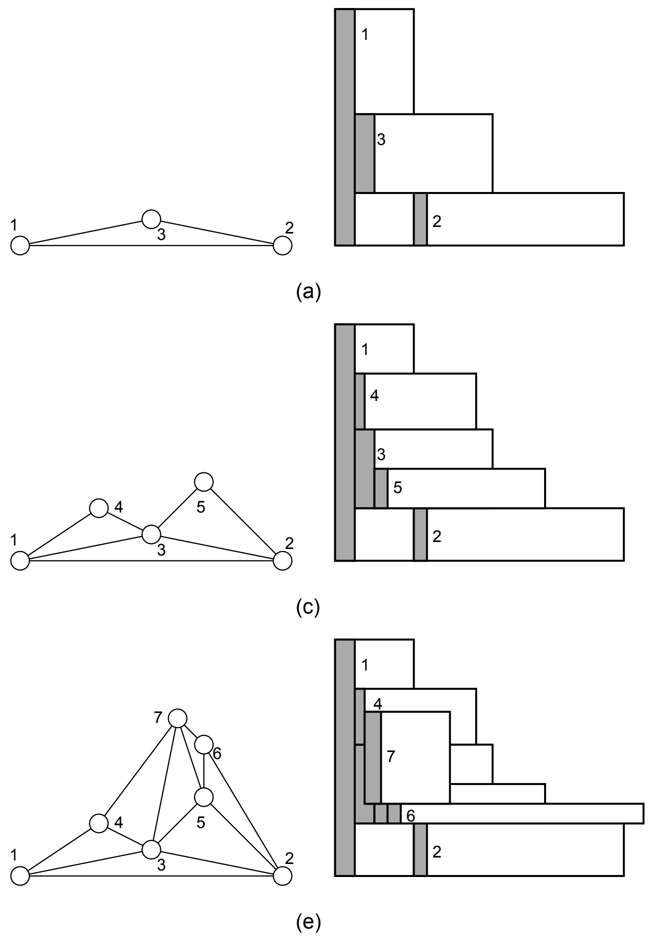

Fig. 3. (a-f) Steps of the algorithm for constructing a contact representation of a planar graph with normal $\mathcal{L}$'s.

neighbors. We then construct the base B and the pillar P of $\mathcal{L}$ for v_i, such that the following conditions holds:

(1) the left side of B touches the right side of the pillar of v_p;
(2) the back side of B touches the front side of v_q's pillar;
(3) the y-coordinate of the bottom side of B is i;
(4) the front side of B has smaller y-coordinate than the front side of the pillar for v_p;
(5) B has unit height;
(6) the bottom side of P is completely contained in the top side of B and the left boundary of P's bottom side coincides exactly with the left boundary of B's top side;
(7) the z-coordinate of the top side of P is k, where k is the canonical number of the highest numbered neighbor of v_i;
(8) the right side of P has smaller x-coordinate than the right side of the pillar for v_q.

Note that for each neighbor v of v_i in G_i, other than v_p and v_q, v_i is the highest numbered neighbor; hence, the pillar of v has its top side at z-coordinate equal to i. This fact, together with the staircase profile for G_{i-1}, ensures that the bottom side of B touches the top side of the pillars for all such vertices. Furthermore, conditions 4 and 8 ensure that the staircase profile is maintained for G_i as well; see Fig. 2(c). Note that we do not restrict the right side of the base. In fact, the right side of the base is visible from the infinite x-coordinate, and for any choice of the x-coordinate for the right side of the base (greater than its left side), the contacts that this $\mathcal{L}$ represents remain unchanged. Call this the *right-extensibility property*. Fig. 3 illustrates the construction of a contact representation of a planar graph with normal $\mathcal{L}$'s.

As a result we now have a (non-proportional) contact representation Γ of G. We modify Γ so that for each vertex v its $\mathcal{L}$ realizes $w(v)$ by volume. First scale down Γ along the y-coordinate so that for each vertex v of G, its $\mathcal{L}$ has volume at most $w(v)$. Now in order to realize the weights, we only need to increase the volumes of $\mathcal{L}$'s. By the right-extensibility property, we can realize the correct volume for each $\mathcal{L}$ by extending its base in the positive x-direction. ■

Next we consider proportional contact representations of the complete graph K_n and the complete bipartite graph $K_{n,m}$.

B. Dense Graphs

Lemma 1: Let $G = (V, E)$ be a complete graph and let $w : V \to \mathbb{R}^+$ be a weight function. Then G has a proportional contact representation with respect to w with broken $\mathcal{L}$'s.

Proof: Call a cuboid C *horizontal* if its length in the x-axis is more than its length in the y-axis; otherwise call C *vertical*. Take an arbitrary order $v_1, \ldots, v_n$, of the vertices

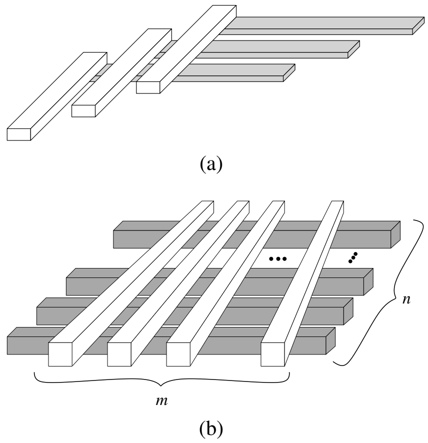

Fig. 4. Proportional contact representations of (a) a complete graph K_n and (b) a complete bipartite graph $K_{n,m}$.

of G. Realize each vertex v_i with a broken $\mathcal{L}$ consisting of a horizontal and a vertical cuboid; see Fig 4(a). The construction has the following properties:

- all the horizontal cuboids are coplanar in the face with the largest z-coordinate;
- all the vertical cuboids are coplanar in the face with the smallest z-coordinate;
- the horizontal (resp., vertical) cuboids for $v_1, \dots, v_n$ are arranged in decreasing order of y-coordinates (resp., x-coordinates);
- the vertical cuboid of v_i touches the horizontal cuboids for $v_{i+1}, \dots, v_n$ for all $i \in \{1, \dots, n-1\}$.

It is easy to see that volumes can be made proportional independently for every vertex, for example, by increasing or decreasing the height of its vertical cuboid. ■

Lemma 2: Let $G = (V, E)$ be a complete bipartite graph and let $w : V \to \mathbb{R}^+$ be a weight function. Then G has a proportional contact representation with respect to w with cuboids.

Proof: Call the two partitions of the vertices A and B. Define *horizontal* and *vertical* cuboids as in the proof of Lemma 1. Realize each vertex of A by a horizontal cuboid and each vertex of B by a vertical cuboid such that the face with the largest z-coordinate for each horizontal cuboid and the face with the smallest z-coordinate for each vertical cuboid are coplanar and in contact; see Fig. 4(b). The volume can be easily made proportional by changing the heights of the cuboids. ■

Lemmas 1 and 2 show that with broken $\mathcal{L}$'s and cuboids one can represent graphs with very high density. On the other hand, sparse graphs (e.g., planar) can also be represented by $\mathcal{L}$'s. Hence, a natural question is whether there exists an appropriate low-complexity polyhedron for representing *all* graphs? The following lemma negatively answers the question.

Lemma 3: There exist infinitely many (bipartite) graphs that have no contact representations in 3D with axis-aligned polyhedra of constant complexity.

Proof: We show that one cannot represent all graphs as contacts of cuboids. Similar arguments work for the case of other axis-aligned polyhedra of constant complexity.

Consider a contact representation Γ of an n-vertex graph G. Assume that the cuboid $[x_i, X_i] \times [y_i, Y_i] \times [z_i, Z_i]$ represents the vertex i in Γ. Then the rectangles for two vertices i and j make a left-right contact (along the x-direction) if $|[y_i, Y_i] \cap [y_j, Y_j]| > 1$, $|[z_i, Z_i] \cap [z_j, Z_j]| > 1$ and either $x_i = X_j$ or $X_i = x_j$. Similar conditions can be derived for front-back contacts (along the y-direction) and top-bottom contacts (along the z-direction). We now show that Γ can be modified to another contact representation $\hat{\Gamma}$ of G in which every (x-, y- and z-) coordinate of the corners of cuboids in $\hat{\Gamma}$ is an integer in the range $[1, 2n]$. Define the cuboids for vertex i in $\hat{\Gamma}$ to be $[\hat{x_i}, \hat{X_i}] \times [\hat{y_i}, \hat{Y_i}] \times [\hat{z_i}, \hat{Z_i}]$. Here for each x-coordinate of a corner of a cuboid in Γ, $\hat{x}$ denotes the rank of the coordinate in the sorted list of all x-coordinates of cuboid corners in Γ. Similarly, define $\hat{y}$ and $\hat{z}$. It is easy to see that rectangles for vertices i and j have a left-right (resp., front-back or top-bottom) contact in Γ if and only if the rectangles for i and j have a left-right (resp., front-back or top-bottom) contact in Γ. Since there are only $2n$ different x-coordinates, $2n$ different y-coordinates and $2n$ different z-coordinates for the corners of the cuboids in $\hat{\Gamma}$, every coordinate can be represented with $O(\log n)$ bits. Thus, only $n \times 4 \times O(\log n)$ = $O(n \log n)$ bits are required to represent $\hat{\Gamma}$ and hence G completely. On the other hand, with $O(n \log n)$ bits one can represent only $O(2^{n \log n})$ different graphs, while there are $\Theta(2^n)$ distinct labeled graphs with n vertices. Therefore, for sufficiently large value of n, one cannot represent all n-vertex graphs with contact representations using cuboids.

A similar argument can be used to prove the lemma for bipartite graphs. Furthermore, it is easy to see that in order to be able to represent all (bipartite) graphs with n vertices, one needs to use objects with complexity $\Theta(n/\log n)$. ■

III. Edge-Weighted Representations

In this section, we consider representations in which the surface area of a contact between two polyhedra realizes given weights. It turns out that normal $\mathcal{L}$'s are sufficient for outerplanar graphs.

Theorem 2: Let $G = (V, E)$ be an outerplanar graph and let $w : E \to \mathbb{R}^+$ be a weight function. Then G has a proportional contact representation with respect to w with normal $\mathcal{L}$'s.

Proof: It suffices to prove the claim for maximal outerplanar graphs. Our proof is by induction on the number of vertices in the given graph G, assuming that G is embedded. If G contains a single edge, then a desired representation Γ is straightforward. Thus we assume that G has at least three vertices. Let $(a, b) \in E$ be an edge on the outer-face of G, and let $c \in V$ be the (unique) common neighbor of a and b. Then G can be split into two subgraphs G_1 and G_2. The subgraph G_1 is induced by the vertices between a and c in counterclockwise order around the outer-face of G, and G_2 is induced by the vertices between b and c in clockwise order around the outer-face; see Fig. 5(a).

In the representation Γ, vertex a is a vertical cuboid of unit height and b is a horizontal one, while all other vertices of G are normal $\mathcal{L}$'s lying in the box bounded by a and b;

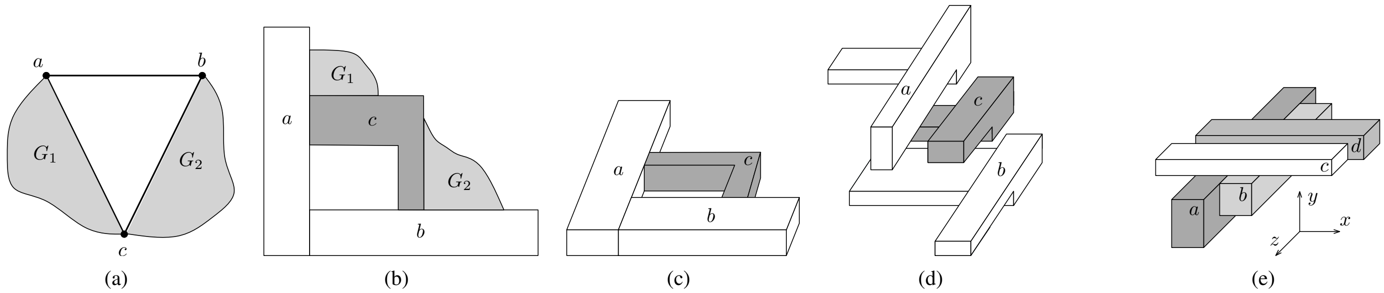

Fig. 5. (a-d) Contact representation of outerplanar graphs: (a) partitioning into G_1 and G_2, (b) recursive step, (c) edge-weighted representation with normal $\mathcal{L}$'s, (d) vertex-edge-weighted representation with broken $\mathcal{L}$'s. (e) A representation of K_4 with cuboids.

see Fig. 5(b) and Fig. 5(c). We call a and b the *base* of Γ. By induction hypothesis, there is a representation Γ_1 for G_1 with base a and c and a representation Γ_2 for G_2 with base c and$b$. Note that the contact between a and c in Γ_1 already realizes the correct area, $w(a, c)$; similarly, the contact between c and b in Γ_2 has area $w(c, b)$. It is easy to see that Γ can be formed by combining Γ_1 and Γ_2 together. To this end, merge the two cuboids corresponding to c in Γ_1 and Γ_2 into a normal $\mathcal{L}$ and extend the cuboids for a and b until they touch; see Fig. 5(c). The area of the contact between a and b can be made $w(a, b)$, which gives the desired representation. ∎

IV. Vertex-Edge-Weighted Representations

In this section, we assume that weights on vertices and edges are given. The goal is to simultaneously realize vertex-weights via volumes and edge-weights via contact areas. We identify several classes of graphs admitting such a representation for any given weights.

Theorem 3: Let $G = (V, E)$ be a planar bipartite graph and let $w : V \cup E \to \mathbb{R}^+$ be a weight function. Then G has a proportional representation with respect to w with cuboids.

Proof: Our construction is based on a segment contact representation (in 2D) of a given graph. In such a representation, vertices are represented by straight-line segments so that no two of them have an interior point in common and two segments touch if and only if the corresponding vertices are adjacent in the graph. It is known that any planar bipartite graph has a segment contact representation in which all the segments are either *vertical* or *horizontal* [9].

We start with the representation of G and modify it so that every segment is a thin rectangle; see Fig. 6(b). We then convert all the rectangles into cuboids with unit heights and lift the horizontal cuboids so that they lie on top of vertical ones, ensuring a small overlap between the top faces of vertical and the bottom faces of horizontal cuboids; see Fig. 6(c). We can now scale up the representation to make all contact areas bigger than required. Consider a horizontal cuboid u touching a vertical cuboid v in its interior to realize the edge $(u, v) \in E$ (for example, $u = c$ and $v = 3$ in Fig. 6(c)). The area of the contact can be realized by reducing (if necessary) the width of u. Similarly, one can realize the required areas for all contacts. It is now easy to see that for any cuboid, we can assign arbitrary volume by changing its height without affecting the area of contacts. ∎

Does any planar graph admit such a proportional contact representation with cuboids? The next lemma negatively answers the question.

Lemma 4: Let $G = (V, E)$ be a graph containing K_4 as a subgraph. Then, there exists a function $w : V \cup E \to \mathbb{R}^+$ such that a vertex-edge-weighted proportional representation with cuboids cannot be constructed.

Proof: Let $w(e) = 1$ for all $e \in E$ and $w(v) = 1 - \epsilon$ for all $v \in V$ for some $0 < \epsilon < 1$. We prove that K_4 has no proportional representation with respect to w with cuboids.

Suppose, for a contradiction, that such a representation exists. Let $V = \{a, b, c, d\}$ and denote by x_v (y_v, z_v, resp.) the length (width, height, resp.) of the cuboid $v \in V$. We consider two cases. First, assume that one of the cuboids, say a, uses at least three (mutually adjacent) faces for contacts with other cuboids. Then $x_a y_a \geq 1$, $x_a z_a \geq 1$, and $y_a z_a \geq 1$ since the area of each such face is at least the edge weight required for that contact. Hence, $(x_a y_a z_a)^2 \geq 1$ and the volume of a is at least 1, which contradicts with the given weight $w(a) < 1$.

Second, assume that none of the cuboids uses three faces for making contacts. It is easy to see that in the representation, the top faces of two cuboids (say, a and b) are coplanar with the bottom faces of two other cuboids (say, c and d); see Fig. 5(e). Using the above argument for the contact between a and b, we have $x_a y_a \geq 1$ and $y_a z_a \geq 1$. Since the volume $w(a) = 1$ is fixed, $x_a y_a z_a < 1$, and hence (i): $x_a < 1$. The contact between a and c gives $x_a y_c \geq 1$; together with (i) it yields (ii): $y_c > 1$. The contact between c and d gives $x_c z_c \geq 1$; combining with (ii), we get $x_c y_c z_c > 1$, contradicting to $w(c) < 1$. ∎

If we consider polyhedra with higher complexity, then outerplanar graphs can be always represented proportionally.

Lemma 5: Let $G = (V, E)$ be an outerplanar graph and let $w : V \cup E \to \mathbb{R}^+$ be a weight function. Then G has a proportional representation with respect to w with broken $\mathcal{L}$'s.

Proof: We modify a representation of Theorem 2. We start with a 2D representation of the outerplanar graph G; see Fig. 5(b). Then vertices are converted into broken $\mathcal{L}$'s consisting of vertical and horizontal cuboids. For every edge $(u, v) \in E$, we make sure that the vertical cuboid of u has an overlap with the horizontal cuboid of v; see Fig. 5(d). The volumes of the broken $\mathcal{L}$'s can be realized by changing the heights of the vertical cuboids, and contact areas can be realized by decreasing the corresponding overlaps between vertical and horizontal cuboids, similar to the proof of Theorem 3. ∎

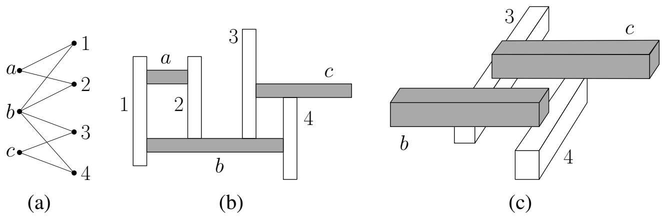

Fig. 6. (a) A planar bipartite graph, (b) its segment contact representation, and (c) (a part of) its representation in 3D with cuboids.

V. Conclusion and Open Problems

We studied proportional contact representations of graphs with axis-aligned polyhedra in 3D. We showed that for some classes of graphs, there always exists a proportional representation with polyhedra consisting of at most two cuboids. On the other hand, not all graphs, not even all bipartite graphs, admit a contact representation with axis-aligned polyhedra with constant complexity. A natural future direction is to improve the polyhedral complexity of the objects needed to represent various classes of graphs. We are particularly interested in the following open problems.

1) For any planar graph, an unweighted contact representation in 3D requires only cuboids. Does the same hold for vertex-weighted proportional representations, or is there an example where some vertices must be represented by union of two cuboids?
2) What graphs have a vertex-edge-weighted representation with cuboids? We showed that planar bipartite graphs always can be represented, while graphs containing K_4 may not admit such a representation. Does every K_4-free planar graph have such a contact representation?
3) While contact representations in 2D directly correspond to planar graphs, one can go beyond planarity in 3D, even with cuboids. For example, complete bipartite graphs can always be represented with cuboids, but K_5 has no representation with cuboids. Is there a good characterization of the class of graphs admitting a contact representation (not necessarily proportional) with cuboids in 3D?
4) Representing graphs with contacts of constant-complexity 3D shapes, such as $\mathcal{L}$'s, is open for graph classes with linear number of edges, such as 1-planar graphs, quasi-planar graphs and other nearly planar graphs.
5) Finally, there are several problems about contact representations in 3D, with non-axis-aligned polyhedra (such as pyramids or tetrahedra). Alam et al. [2] have recently found some classes of graphs, such as complete bipartite and tripartite graphs, that can be represented by non-proper contact of tetrahedra. The characterization and recognition of general and planar graphs representable by contacts of tetrahedra is an interesting open problem.

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