Simultaneous Embedding of Planar Graphs*

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Abstract

Simultaneous embedding is concerned with simultaneously representing a series of graphs sharing some or all vertices. This forms the basis for the visualization of dynamic graphs and thus is an important field of research. Recently there has been a great deal of work investigating simultaneous embedding problems both from a theoretical and a practical point of view. We survey recent work on this topic.

1 Introduction

Traditional problems in graph drawing involve the layout of a single graph, whereas in simultaneous graph drawing we are concerned with the layout of multiple related graphs. In particular, consider the problem of drawing a series of graphs that share all, or parts of the same vertex set. The graphs may represent different relations between the same set of objects, or alternatively, the graphs may be the result of a single relation that changes through time. In this work we survey efforts to address the following problem: Given a series of graphs that share all, or parts of the same vertex set, what is a natural way to layout and display them? The layout and display of the graphs are different aspects of the problem, but also closely related, as a particular layout algorithm is likely to be matched best with a specific visualization technique. As stated above, however, the problem is too general and it is unlikely that one particular layout algorithm will be best for all possible scenarios. Consider the case where we only have a pair of graphs in the series, and the case where we have hundreds of related graphs. The “best” way to layout and display the two series is likely going to be different. Similarly, if the graphs in the sequence are very closely related or not related at all, different layout and display techniques may be more appropriate.

For the layout of the graphs, there are two important criteria to consider: the \textit{readability} of the individual layouts and the \textit{mental map preservation} in the series of drawings. The readability of individual drawings depends on aesthetic criteria such as display of symmetries, uniform edge lengths, and minimal number of crossings. Preservation of the mental map can be achieved by ensuring that vertices that appear in consecutive graphs in the series, remain in the same positions. These two criteria are often contradictory. If we individually layout each graph, without regard to other graphs in the series, we may optimize readability at the expense of mental map preservation. Conversely, if we fix the vertex positions in all graphs, we are optimizing the mental map preservation but the individual layouts may be far from readable. In simultaneous graph embedding, vertices are placed in the exact same locations in all the graphs, while the layout of the edges may differ.

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Visualization of related graphs, that is, graphs that are defined on the same set of vertices, arise in many different settings. Software engineering, databases, and social network analysis, are all examples of areas where multiple relationships on the same set of objects are often studied. In evolutionary biology, phylogenetic trees are used to visualize the ancestral relationship among groups of species. Depending on the assumptions made, different algorithms produce different phylogenetic trees. Comparing the outputs and determining the most likely evolutionary hypothesis can be difficult if the drawings of the trees are laid out independently of each other.

While in some of the above examples the graphs are not necessarily planar, solving the planar case can provide intuition and ideas for the more general case. With this in mind, here we concentrate on the problem of simultaneous embedding of planar graphs. Simultaneous embedding of planar graphs generalizes the notion of traditional graph planarity and is motivated by its relationship with problems of graph thickness, geometric thickness, and applications such as the visualization of graphs that evolve through time.

The thickness of a graph is the minimum number of planar subgraphs into which the edges of the graph can be partitioned; see [51] for a survey. Thickness is an important concept in VLSI design, since a graph of thickness $k$ can be embedded in $k$ layers, with any two edges drawn in the same layer intersecting only at a common vertex and vertices placed in the same location in all layers. A related graph property is geometric thickness, defined to be the minimum number of layers for which a drawing of $G$ exists having all edges drawn as straight-line segments [22]. Finally, the book thickness of a graph $G$ is the minimum number of layers for which a drawing of $G$ exists, in which edges are drawn as straight-line segments and vertices are in convex position [7]. It has been shown that the book thickness of planar graphs is no greater than four [54].

**Problem definitions.** This paper is structured along three basic simultaneous embedding results for planar graphs, Simultaneous Geometric Embedding (SGE), Simultaneous Embedding with Fixed Edges (SEFE), and Simultaneous Embedding (SE). For all three problems the input always consists of two planar graphs $G^\oplus = (V^\oplus, E^\oplus)$ and $G^\otimes = (V^\otimes, E^\otimes)$ sharing a common subgraph $G = (V, E) = (V^\oplus \cap V^\otimes, E^\oplus \cap E^\otimes)$. The most strict variant is Simultaneous Geometric Embedding (SGE), which asks for planar straight-line drawings of $G^\oplus$ and $G^\otimes$, such that common vertices have the same coordinates in both drawings. The requirements of SGE are very strict, and as we will see in Section 2 there exist a lot of examples that do not admit such an embedding. While the problem Simultaneous Embedding with Fixed Edges still requires common vertices to have the same coordinates, it relaxes the straight-line requirement by allowing arbitrary curves for representing edges. To maintain the mental map, common edges are still required to be represented by the same curves. Finally, Simultaneous Embedding drops the constraints on the curves altogether and just requires common vertices to have the same coordinates. For all these problems it is common to also use the problem name to denote a corresponding embedding, that is we also say that $G^\oplus$ and $G^\otimes$ have an SGE, SEFE or SE if they admit solutions to these problems. Moreover, all these problems readily generalize to $k > 2$ input graphs $G^\oplus, \ldots, G^\otimes$, by requiring that the conditions hold for each pair of graphs. In this case a common restriction is to require that all input graphs share exactly the same graph $G$, that is $G = G^\oplus \cap G^\otimes$ for $i \neq j$. We call this behavior sunflower intersection.

We note that simultaneous embedding problems are closely related to constrained embedding problems. For example if the planar embedding of one of the two graphs of an instance of SEFE is already fixed, the problem of finding a SEFE is equivalent to finding an embedding of the second graph respecting a prescribed embedding for a subgraph, namely the common graph. This constrained embedding problem is known as Partially Embedded Planarity. Angelini et al. [2] show that this problem can be solved in linear time and, in the spirit of Kuratowski’s theorem, Jelínek et al. [46] characterize the yes-instances by forbidden substructures. A similar tie to constrained embedding problems exists in the case of SE. After fixing the drawing of
one of the two input graphs it remains to draw a single graph without crossings at prescribed vertex positions. This problem is known as **Point Set Embedding** and Pach and Wenger show that this is always possible [52]. There are other, less obvious relations between simultaneous embedding and constrained embedding problems, which will be described later.

**Overview and Outline.** This work starts with the three simultaneous embedding problems SGE, SEFE, and SE, and we discuss each of them in one of the following sections. There are three major classes of results on simultaneous embedding problems. The first class contains algorithms that, for given graphs with certain properties, always produce a simultaneous embedding, perhaps with additional quality guarantees. These results show the existence of simultaneous embeddings for the corresponding graph classes. The second class contains counterexamples that do not admit a simultaneous embedding. The third class contains algorithms and complexity results for the problem of testing whether a given instance admits a simultaneous embedding.

We present a survey of the results on SGE in Section 2. Due to the strong requirements of SGE results of the first type, which identify classes of graphs that always admit a simultaneous embedding, exist only for very few and strongly restricted graph classes. For example, even a path and a tree of depth 4 may not have an SGE [4]. Moreover, it is NP-hard to decide SGE and there are no further results of the third type, that is algorithms testing whether an instance has an SGE or not, even for restricted instances.

Section 3 presents the SEFE problem, which turns out to be much less restrictive than SGE. For example, a tree and a path do always admit a SEFE although they do not have an SGE [33]. On the other hand, examples not having a SEFE are also counterexamples for SGE. Moreover, it is still open whether SEFE can be tested in polynomial time for two graphs, whereas it is NP-complete for three or more graphs [35]. However, for two graphs, there exist several results of the third type, that is testing algorithms, for restricted inputs. For example, it is possible to decide in linear time whether a pair of graphs admits a SEFE or not, if the common graph is biconnected [3, 40].

In Section 4, we consider the least restrictive simultaneous embedding problem, SE, which only requires common vertices to have the same coordinates in all drawings. As every planar graph can be drawn without crossings even if the position of every vertex is fixed [52], there are no counterexamples for SE and it is not necessary to have a testing algorithm. The results on SE focus on creating simultaneous embeddings such that edges have few bends and the resulting drawings use small area.

Sections 5–8 present several variants of approaches to simultaneous embedding that do not quite fall into the categories of the three main problems. The problem variants discussed in Section 5 relax the requirement of having a fixed mapping between the vertices of \( G^1 \) and \( G^2 \). They rather ask whether a suitable mapping can be found such that a SEFE exists [11]. Colored SGEs are somewhere between and allow the mapping to identify only vertices having the same color [10]. Section 6 deals with matched drawings requiring straight-line drawings of the two input graph such that each common vertex has only the same \( y \)-coordinate in both drawings. Other work, discussed in Section 7, deals with the problem of simultaneously representing a planar graph and its dual [54] and considers different types of simultaneous representations, such as simultaneous intersection representations, as introduced by Jampani and Lubiw [44]. Section 8 presents several practical approaches to simultaneous embedding problems.

Finally, in Section 9, we present a list of open questions. The list contains questions that have been open for several years, as well as questions that are motivated by recent research results.
Figure 1: The union of the graph on the left and the graph on the right is a $K_5$, but the middle drawing shows a simultaneous geometric embedding of the two graphs.

2 Simultaneous Geometric Embedding

In this section we consider the most desirable (and most restrictive) kind of simultaneous drawings, the SGEs. Most results on that problem are summarized in Figure 2. Before we describe these results in more detail we start with a small example. While it may be tempting to say that if the union of two graphs contains a subdivision of $K_5$ or $K_{3,3}$ then the two graphs have no simultaneous geometric embedding, this is not the case; see Figure 1. In fact, while planarity testing for a single graph can be done in linear time [43], Estrella-Balderrama et al. [28] show that the decision problem SGE is NP-hard. Other results concerning the complexity of SGE (for example for restricted graph classes) are not known.

In the following we describe the results illustrated in Figure 2. We start with algorithms always creating an SGE for the case that the input is restricted to special graph classes. We then continue with graph classes containing counterexamples. Finally, we consider the results not fitting in one of these two cases.

Brass et al. [11] give several algorithms for different restricted graph classes always creating an SGE. In the simplest case $G^\oplus_1$ and $G^\oplus_2$ are both required to be paths. This result is easy to prove and also provides good intuition for most of the positive results:

**Theorem 1.** For two paths $P_1^\oplus$ and $P_2^\oplus$ on the same vertex set $V$ of size $n$ an SGE on a grid of size $n \times n$ can be found in linear time.

*Proof.* For each vertex $u \in V$, we embed $u$ at the integer grid point $(p_1^\oplus, p_2^\oplus)$, where $p_i^\oplus \in \{1, 2, \ldots, n\}$ is the vertex’s position in the path $P_i^\oplus$, $i \in \{1, 2\}$. Then, $P^\oplus$ is embedded as an $x$-monotone polygonal chain, and $P^\oplus_2$ is embedded as a $y$-monotone chain. Thus, neither path is self-intersecting; see Figure 3.

Brass et al. [11] also consider more general graph classes, such as caterpillars (trees being paths after the removal of all leaves), stars (trees with at most one inner vertex called center), and extended stars (collection of stars with an additional special root and paths from the special root to the centers of all stars). They show that a caterpillar and a path admit an SGE on a grid of size $n \times 2n$, which can be extended to two caterpillars on a grid of size $3n \times 3n$. Moreover, they can simultaneously embed two stars on a $3 \times (n - 2)$ grid and extend it to the case of $k$ stars on an $O(n) \times O(n)$-grid. Finally, the pairs path plus extended star and cycle plus cycle can be embedded on $O(n^2) \times O(n)$ and $4n \times 4n$ grids, respectively. The latter two results both extend the case of two paths (when neglecting the grid size).

The result for two cycles was further extended by Duncan et al. [23] and Cabello et al. [13]. Duncan et al. [23] show that a graph with maximum degree 4 has geometric thickness 2. To this end, they show that two graphs with maximum degree 2 always admit a simultaneous geometric embedding. However, their algorithm computes drawings with potentially large area. Cabello et al. [13] show the existence of an SGE for a wheel (union of a star and a cycle on its leaves) and a cycle. They moreover give algorithms for the pairs tree plus matching (graph with maximum degree 1) and outerpath (outerplanar graph whose weak dual is a path) plus matching. The former algorithm uses only two slope for the matching edges, for the latter one slope suffices.
Given a planar graph and a path on the same vertices, the order of the vertices in the path induces a layering on the vertices. Cappos et al. [14] give a linear-time algorithm that computes an SGE of a planar graph and a path if the planar graph is level-planar with respect to the layering induced by the path. Angelini et al. [4] show that every tree of depth 2 has an SGE.

In contrast to the positive results, Brass et al. [11] give several examples not admitting an SGE. They show the existence of two planar graph without a simultaneous embedding and extended this result to two outerplanar graphs. Two results we present in more detail are the counterexample for a planar graph and a path by Brass et al. [11] and Erten and Kobourov [24] and the counterexample of three paths by Brass et al. [11].

**Theorem 2.** There exists a planar graph $G$ and a path $P$ not admitting an SGE.

**Proof.** Consider the graph $G$ and the path $P$ as shown in Figure 4. Let $G'$ be the subgraph of $G$ induced on vertices $\{1, 2, 3, 4, 5\}$, and $G''$ be the subgraph of $G$ induced on vertices $\{2, 6, 7, 8, 9\}$. 

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Figure 2: Overview over the so far known results on SGE. Each box represents one result and an arrow highlights that the source-result is extended by the target-result. The arrowheads are empty for the cases in which this is only true if the grid size is neglected. Note that transitive arrows are omitted.
Figure 3: Two paths simultaneously embedded such that one path is \( x \)-monotone and the other is \( y \)-monotone.

Figure 4: A planar graph \( G \) and a path \( P \) that do not allow an SGE.

Since \( G \) is triconnected fixing the outer face fixes an embedding for \( G \). With the given outer face of \( G \), the path \( P \) contains two crossings: one involving \((2, 4)\), and the other one involving \((6, 8)\). Graph \( G' \) has six faces and unless we change the outer face of \( G' \) such that it contains the edge \((1, 3)\) or \((3, 5)\), the edge \((2, 4)\) is involved in a crossing in the path. Similarly for \( G'' \), unless we change its outer face such that it contains \((2, 7)\) or \((7, 9)\), the edge \((6, 8)\) is involved in a crossing in the path. However \( G' \) and \( G'' \) do not share any faces and removing both crossings depends on taking two different outer faces, which is impossible. Thus, regardless of the choice for the outer face of \( G \), path \( P \) contains a crossing.

\[ \text{Theorem 3. There exist three paths } P_1, P_2 \text{ and } P_3 \text{ not admitting an SGE.} \]

\[ \text{Proof. A path of } n \text{ vertices is simply an ordered sequence of } n \text{ numbers. The three paths we consider are: } 714269358, 824357169 \text{ and } 758261439. \text{ For example, the sequence } 714269358 \text{ represents the path } (v_7, v_1, v_4, v_2, v_6, v_9, v_3, v_5, v_8). \text{ We will write } ij \text{ for the edge connecting } v_i \text{ to } v_j. \text{ The union of these paths contain the following twelve edges.

\[ E = \{14, 16, 17, 24, 26, 28, 34, 35, 39, 57, 58, 69\} \]

It is easy to see that the graph \( G \) consisting of these edges is a subdivision of \( K_{3,3} \) and therefore non-planar: collapsing 1 and 7, 2 and 8, 3 and 9 yields the classes \{1,2,3\} and \{4,5,6\}.

It follows that there are two nonadjacent edges of \( G \) that cross each other. It is easy to check that every pair of nonadjacent edges from \( E \) appears in at least one of the paths given above. Therefore, at least one path will cross itself which completes the proof.

Cabello et al. [13] extend the counterexample for the case that \( G^\oplus \) is a path to the case where \( G^\oplus \) is a matching. Moreover, they give an example of six matchings not admitting an SGE. Note that this does not directly follow by dividing three paths without an SGE into six matchings,
as the resulting matchings allow crossings that were not allowed before. Another extension of the case where $G^\oplus$ is a path was given by Frati et al. [34] who give a counterexample where $G^\oplus$ is a path and $G$ is a set of isolated vertices, that is $G^\oplus$ and $G$ are edge disjoint.

The question of whether two trees always admit an SGE was open for several years, before it was answered in the negative by Geyer et al. [37] with a construction involving two very large trees. This of course extends the result of two outerplanar graphs not having an SGE by Brass et al. [11]. Angelini et al. [4] further extended it to the case of a tree and a path without an SGE. More precisely, they give an example of a tree of depth 4 and an edge disjoint path not having an SGE. Recall that a tree of depth 2 does always admit a simultaneous embedding with a path, thus in this case the gap between positive and negative results is quite small.

Frati et al. [34] consider the restricted case where each input graph has a prescribed combinatorial embedding. They show that the pair path plus star admits an SGE even if the embedding of the star is fixed. They can extend this result to a double-star (tree with up to two inner vertices) if it is edge disjoint to the path. On the other hand they show that fixing the embedding of two caterpillars may lead to an counterexample, whereas they admit an SGE if the embedding is not fixed. Another counterexample is the pair outerplanar graph with fixed embedding plus edge-disjoint path.

An interesting additional restriction to SGEs was considered by Argyriou et al. [6, 5], combining SGE with the RAC drawing convention (RAC – Right-Angular Crossing). They try to find an SGE such that crossings between exclusive edges of different graphs are restricted to right-angular crossings. Argyriou et al. consider only the case where the edge sets of both graphs are disjoint. They present one negative and one positive result for this problem. The negative result consists of a wheel and a cycle not admitting an SGE with right-angular crossings. On the other hand they show the existence of such a drawing on a small integer grid for the case that one of the graphs is a path or a cycle and the other is a matching. Moreover, they give a linear-time algorithm to compute such a drawing.

3 Simultaneous Embedding with Fixed Edges

Figure 5 illustrates the results on the problem SEFE classified in the three categories described before.

We start with instances that are known to always have a SEFE. Erten and Kobourov [24] show that a tree and a path can always be embedded simultaneously. They additionally give an algorithm finding a simultaneous embedding in $O(n)$ time on a grid of size $O(n) \times O(n^2)$ such that the edges of $G^\oplus$ and $G^\otimes$ have at most one and zero bends per edge, respectively. Note that a grid of size $O(n^2) \times O(n^2)$ is necessary if the bends are required to be drawn on grid points. Di Giacomo and Liotta [21] extend this result to the case of an outerplanar graph and a path with the same grid and bend requirements. They extend it further to the case where $G^\oplus$ and $G^\otimes$ are outerplanar and the common graph $G$ is a collection of paths and to the case where $G^\oplus$ is outerplanar and $G^\otimes$ is a cycle. However, in both cases a grid of size $O(n^2) \times O(n^2)$ and up to one bend per edge are required. If the grid and bend requirements are completely neglected, the results considering the pairs tree plus path and outerplanar graph plus path can be extended to the case where one of the two graphs is a tree. Frati [33] shows how a tree $G^\oplus$ can be simultaneously embedded with an arbitrary planar graph $G^\otimes$. This algorithm still works if $G^\oplus$ contains one additional edge that is not a common edge, yielding the result that every graph with at most one cycle (a pseudoforest) can be embedded simultaneously with every other planar graph if the common graph does not contain this cycle. Fowler et al. [30] extend this result further to the case where $G^\oplus$ contains only disjoint cycles and the common graph $G$ does not contain a cycle.

Aside from instances always having a SEFE, there are also examples that cannot be simultaneously embedded. Brass et al. [11] give examples for $k$ outerplanar graphs, three paths and an
Figure 5: Overview over the so far known results on SEFE. Each box represents one result and an arrow highlights that the source-result is extended by the target-result. The arrowheads are empty for the cases in which this is only true, if the number of bends per edge, the consumed grid size or the necessary running time is neglected. Note that transitive arrows are omitted.
outerplanar graph plus a planar graph not having a SEFE. The results concerning outerplanar graphs can be extended to the case where both graphs are outerplanar [33]. In between the positive and negative results there are some characterizations stating which instances have a SEFE and which do possibly not. Fowler et al. [31] give a characterization of the graphs $G^\odot$ having a SEFE with every other planar graph. This of course extends all results concerning only $G^\odot$.

In particular, the results that a tree can be simultaneously embedded with every other graph, whereas an outerplanar graph cannot, are extended. This characterization essentially requires that $G^\odot$ must not contain a subgraph homeomorphic to $K_3$ (a triangle) and an edge not attached to this $K_3$, see Figure 6 for an example. The considerations made for this characterization additionally yield a characterization for the outerplanar graphs $G^\odot$ having a simultaneous embedding with every other outerplanar graph $G^\odot$. This of course extends the result that two outerplanar graphs possibly do not have a SEFE. Another characterization, in terms of the common graph, is given by Jünger and Schulz [47]. They show that two graphs can be simultaneously embedded if the common graph $G$ has only two embeddings, whereas in all other cases graphs $G^\odot$ and $G^\odot$ with the common graph $G$ not having a SEFE can be constructed. They additionally show that finding a SEFE is equivalent to finding combinatorial embeddings of $G^\odot$ and $G^\odot$ inducing the same combinatorial embedding, that is the same orders of edges around vertices and the same relative positions of connected components to one another, on the common graph $G$ [47, Theorem 4]. Note that it is not obvious and not even true for more than two graphs [1]. As this result is heavily used in most algorithms solving the decision problem SEFE, we state it as a theorem.

**Theorem 4.** Two graphs $G^\odot$ and $G^\odot$ with common subgraph $G$ admit a SEFE if and only if they admit combinatorial embeddings inducing the same embedding on $G$.

Since SEFE has positive and negative instances, it would be nice to have an algorithm deciding for given graphs, whether they can be embedded simultaneously. If more than two graphs are allowed, this problem is known to be NP-complete [35], whereas the complexity for two graphs is still open. However, there are several results solving SEFE for special cases. For example the characterization of outerplanar graphs having a simultaneous embedding with every other outerplanar graph mentioned above yields a linear-time algorithm for testing whether two outerplanar graphs have a SEFE [31]. Fowler et al. [30] show how to test SEFE, if $G^\odot$ is a pseudoforest, that is a graph with at most one cycle. Note that, as mentioned above, such an instance always has a SEFE if this single cycle is not contained in $G$. This result can be extended to the case where $G^\odot$ contains up to two cycles, if $G$ does not contain the second cycle, that is $G$ is a pseudoforest. To achieve this result the following auxiliary problem was solved. Given a planar graph $G$ with a designated cycle $C$ and a partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of the vertices not contained in $C$, does $G$ admit a planar embedding, such that all vertices in $P_i$ are on the same side of the cycle for every set $P_i$? Note that this again is a constrained embedding problem, showing that constrained and simultaneous embedding are closely related.

Haeupler et al. [10] give a linear-time algorithm to solve SEFE for the case that the common graph is biconnected. Their solution is an extension of the planarity testing algorithm by Haeupler and Tarjan [11]. This planarity testing algorithm starts with a completely unembedded

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**Figure 6:** $G^\odot$ (a) and $G^\odot$ (b) do not admit a SEFE (c) as $G^\odot$ forces the vertices 4 and 5 to different sides of the triangle $\Delta 123$. 

(a) (b) (c)
graph and adds vertices iteratively, such that the unembedded part is always connected, ensuring that it can be assumed to lie in the outer face of all embedded components. While inserting vertices, they keep track of the possible embeddings of the embedded parts by representing the possible orders of half-embedded edges around every component with a PQ-tree having these edges as leaves. In a PQ-tree every inner node is either a Q-node fixing the order of edges incident to it up to a flip or a P-node allowing arbitrary orders. In this way a PQ-tree represents a set of possible orders of its leaves. A completely different approach is used by Angelini et al. [3] to solve SEFE in linear time if the common graph is biconnected. They choose an order for the common graph bottom up in its SPQR-tree such that the private edges can be added. Another approach by Bläsius and Rutter [8] also uses PQ-trees. They use that the possible orders of edges around every vertex of a biconnected planar graph can be represented by a PQ-tree, yielding a set of PQ-trees, one for each vertex. To obtain a planar embedding, the orders for the PQ-trees have to be chosen consistently. Bläsius and Rutter define the problem SIMULTANEOUS PQ-Ordering asking for orders in PQ-trees that are chosen consistently, which can, among other applications, be used to represent all planar embeddings of a biconnected graph. This extends to the case of two biconnected planar graphs enforcing shared edges to be ordered the same and thus yields a quadratic time algorithm for SEFE if $G^\ominus$ and $G^\oplus$ are biconnected and $G$ is connected. The latter requirement comes from the fact that only orders of edges around vertices are taken into account, relative positions of connected components to one another are neglected. Note that this result extends the case where $G$ is biconnected for the following reason. If $G$ is biconnected, then $G$ is completely contained in a single block (maximal biconnected component) of $G^\ominus$ and $G^\oplus$. Thus, even if $G^\ominus$ or $G^\oplus$ are not biconnected, they contain only one block that is of interest, all other blocks can simply be attached to this block. The result by Bläsius and Rutter can be slightly extended to the case where the graphs $G^\ominus$ and $G^\oplus$ contain cut-vertices incident to at most two non-trivial blocks (blocks not consisting of a single edge), including the special case where both graphs have maximum degree 5. The SIMULTANEOUS PQ-ORDERING approach again shows the strong relation between simultaneous and constrained embedding as in an instance of SEFE the two input graphs constrain the possible orders of some of the edges around vertices of one another with PQ-trees.

Angelini et al. [3] show the equivalence between SEFE and a constrained version of the PARTITIONED 2-PAGE BOOK EMBEDDING problem. An instance of PARTITIONED 2-PAGE BOOK EMBEDDING is a graph and a partition of its edges into two subsets. It asks whether all vertices can be arranged on a straight line (the spine) such that each of the edge partitions can be embedded without crossings in one of the two incident half-planes (pages of the book). PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING additionally has a tree as input with the vertices of the graph as leaves. It is then required that the tree admits an embedding such that the order of its leaves is equal to the order of vertices on the spine. In other words, the allowed orders of vertices on the spine is constrained by a PQ-tree containing no Q-nodes. Angelini et al. [3] prove the following theorem and we sketch their proof here.

**Theorem 5.** The problems SEFE for two graphs with connected intersection and PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING have the same time complexity.

**Proof.** Angelini et al. [3] first show that an instance of SEFE where the common graph is connected can be modified (yielding an equivalent instance) such that the common graph is a tree. Moreover, each private edge is incident to leaves of this tree. They then show the equivalence to an instance of PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING where the common graph is the constraining tree, the leaves of this tree are the vertices that need to be placed on the spine and the private edges of each of the graphs is one of the partitions.

In the following we sketch this construction using the example in Figure 7. The instance in (a) having a tree $T$ as common graph such that each private edge is incident to a leaf admits a SEFE. All private edges are embedded outside the dashed cycle around $T$ in (b) containing all its
leaves. Choosing another face as outer face and cutting the cycle at an arbitrary position yields a SEFE where all leaves of $T$ are embedded on a straight line (c) with all private edges on the same side. This directly yields the Partitioned $T$-Coherent 2-Page Book Embedding in (d) of the private edges respecting the tree $T$. This shows the equivalence of SEFE and Partitioned $T$-Coherent 2-Page Book Embedding as the constructions works the same in the opposite direction.

For the restricted case that $T$ is a star, Partitioned $T$-Coherent 2-Page Book Embedding reduces to the problem Partitioned 2-Page Book Embedding that can be solved in linear time [42]. Thus the above result directly implies that SEFE can be solved in linear time if the common graph is a star.

All results mentioned thus far require $G$ to be connected and most results also require $G^\circ$ and $G^{\circ\circ}$ to be connected. Bläsius and Rutter [9] consider the case where this does not hold. They show that it can be assumed without loss of generality that both graphs $G^\circ$ and $G^{\circ\circ}$ are connected. In the case that $G$ is connected, one only has to deal with orders of edges around vertices and can neglect relative positions of connected components to one another. Bläsius and Rutter approach SEFE from the opposite direction, caring only about the relative positions, neglecting the orders of edges around vertices. More precisely, they give a linear-time algorithm solving SEFE if the common graph is a set of disjoint cycles. They can extend this result to a quadratic-time algorithm for the case where $G$ consists of arbitrary connected components, each with a fixed planar embedding. Both results extend to an arbitrary number of graphs with sunflower intersection. Recall that sunflower intersection means that all graphs intersect in the same common subgraph. Moreover, they give a succinct representation of all simultaneous embeddings.

A result not really fitting in one of the three above classes by Duncan et al. [23] considers the restricted case of SEFE where each edge has to be a sequence of horizontal and vertical segments with at most one bend per edge. They show that two graphs with maximum degree 2 always admit such a SEFE on a grid of size $O(n) \times O(n)$ by adapting their linear-time algorithm computing an SGE for these types of graphs (on a larger grid).

Angelini et al. [1] consider the case where the embedding of each of the input graphs is already fixed. With this restriction SEFE becomes trivial for two graphs since it remains to test whether the two graphs induce the same embedding on the common graph. They show that it can also be decided efficiently for three graphs. However, it becomes NP-hard for at least fourteen graphs. They also consider the problem SGE for the case that the embedding of each graph is fixed and show that it is NP-hard for at least thirteen graphs.

4 Simultaneous Embedding

In the most restricted version of the problem, SGE, we insist that vertices are placed in the same position, and edges must be straight-line segments. The SEFE setting relaxes the straight-line
condition but maintains that edges common to multiple graphs are realized the same way in each. In the least restrictive setting, SE, we allow the same edge to be realized differently in different graphs.

It has already been mentioned that simultaneous embedding of multiple graphs can be thought of as a generalization of the notion of planarity. A classical result about planar graphs connects the notion of a planar graph with that of a straight-line, crossing-free drawing thereof. Specifically, Wagner in 1936 [55], Fáry in 1948 [29], and Stein in 1951 [53] independently show that if a graph has a drawing without crossings, using arbitrary curves as edges, then there exists a drawing of the graph also without crossings, but with edges drawn as straight-line segments. For multiple graphs, however, this result does not hold. That is, given several graphs on the same $n$ vertices, we can surely realize each graph without crossings, using arbitrary curves as edges and the same vertex positions for each graph. But (except in very special circumstances such as the positive examples in the Section 2) we cannot guarantee that there exist vertex positions that allow the realization of each graph with straight-line segments and without crossings. If this were true, then the vertex positions would be a universal pointset for graphs on $n$ vertices, and it is known that universal pointsets of linear size do not exist [17].

Pach and Wenger [52] show that every planar graph can be drawn without crossings with a prespecified position for every vertex. Thus, for every pair of planar graphs an SE can be created by drawing the first graph arbitrarily and the second graph to the vertex positions specified by the first drawing. Thus, there are neither negative examples nor is it necessary to have testing algorithms. However, the drawing of the second graph may have linearly many bends per edge, thus it is of interest to find an SE with fewer bends. Erten and Kobourov [24] show that every two graphs can be drawn simultaneously in $O(n^2)$ time with at most three bends per edge on a $O(n^2) \times O(n^2)$ grid ($O(n^3) \times O(n^3)$ if bends need to be placed on grid points), where $n$ is the number of vertices. This result was improved by Di Giacomo and Liotta [20, 21] to at most two bends per edge in general and one bend per edge, if $G^\text{O}$ and $G^\text{O}$ are both sub-Hamiltonian. That is, they can be augmented to become Hamiltonian maintaining planarity, and an augmentation together with a Hamiltonian cycle is given with the input. Similar results were obtained by Kammer [48]. As series-parallel graphs [18], trees and outerplanar graphs [16, 7] are always sub-Hamiltonian and an augmentation together with a Hamiltonian cycle can be computed in linear-time this result yields a linear time algorithm to compute an SE of $G^\text{O}$ and $G^\text{O}$ with one bend per edge on a grid of size $O(n^2) \times O(n^2)$ if each of the graphs $G^\text{O}$ and $G^\text{O}$ is series-parallel, a tree or outerplanar.

Cappos et al. [14] show that a path and an outerplanar graph can be simultaneously embedded in linear time such that edges in the outerplanar graph are straight-line segments and each edge in the path consists of a single circular arc. Alternatively, the path edges may be piecewise linear with at most two bends per edge.

5 Colored Simultaneous Embedding

Since SGE can be too restrictive, various relaxations have been considered. The two relaxed versions already mentioned, SEFE and SE relax the requirement of straight-line edges, and even the requirement that common edges are drawn the same way in both drawings. Another way to relax the constraints of the original SGE problem is to allow changes in vertex positions in different graphs.

Until this point we had assumed that multiple input graphs have labeled vertices and thus the mapping between the vertices of the graphs is part of the input. In simultaneous embedding without mapping we are interested in computing plane drawings for each of the given graphs on the same set of points, where any vertex can be placed at any of the points in the point set. This setting of the problem was investigated in the very first paper on SGE [11] and is the source of one of the longest standing open problems in the area.
Colored Simultaneous Embeddings (CSE), were introduced by Brandes et al. \[10\] and allow us to generalize the problems above so that the versions with and without mappings become special cases. Formally, the problem of CSE is defined as follows. The input is a set of planar graphs \( G_1^\odot = (V, E_1^\odot), G_2^\odot = (V, E_2^\odot), \ldots, G_k^\odot = (V, E_k^\odot) \) on the same vertex set \( V \) and a partition of \( V \) into \( c \) classes, which we refer to as colors. The goal is to find plane straight-line drawings \( D^\odot_i \) of \( G_i^\odot \) using the same \( |V| \) points in the plane for all \( i = 1, \ldots, k \), where vertices mapped to the same point are required to be of the same color. We call such graphs \( c \)-colored graphs. Given the above definition, simultaneous embeddings with and without mapping correspond to colored simultaneous embeddings with \( c = |V| \) and \( c = 1 \), respectively. Thus, when a set of input graphs allows for a simultaneous embedding without mapping but does not allow for a simultaneous embedding with mapping, there must be a threshold for the number of colors beyond which the graphs can no longer be embedded simultaneously.

Colored simultaneous embeddings provide a way to obtain near-simultaneous embeddings, where we place corresponding vertices nearly, but not necessarily exactly, at the same locations. Relaxing the constraint on the size of the pointset allows for a way to more easily obtain near-simultaneous embeddings, where we attempt to place corresponding vertices relatively close to one another in each drawing. For example, if each cluster of points in the plane has a distinct color, then even if a red vertex \( v \) placed at a red point \( p \) in \( G_1^\odot \) has moved to another red point \( q \) in \( G_2^\odot \), the movement is limited to the area covered by the red points. Brandes et al. \[10\] show several positive and negative results about CSE. In particular they show that there exist universal pointsets of size \( n \) for 2-colored paths and spiders as well as 3-colored paths and caterpillars. It is also shown that a 2-colored tree (or even a 2-colored outerplanar graph) and any number of 2-colored paths can be simultaneously embedded. In the negative direction, there exist a 2-colored planar graph and pseudo-forest, three 3-colored outerplanar graphs, four 4-colored pseudo-forests, three 5-colored pseudo-forests, five 5-colored paths, two 6-colored biconnected outerplanar graphs, three 6-colored cycles, four 6-colored paths, and three 9-colored paths that cannot be simultaneously embedded.

Frati et al. \[34\] continue the investigation of near-SGE’s, that is they try to find straight-line drawings of the input graphs with a small distance between every pair of common vertices in different drawings. As a negative result, they present a pair of graphs such that in every pair of drawings there exists a common vertex with distance linear in the size of the input. On the other hand, they present positive results for a sequence of paths and a sequence of trees for the case that every two consecutive graphs in the sequence are similar with respect to a parameter measuring their similarity. It can then be shown that the distance of a common vertex in two consecutive drawings depends linearly on this parameter.

6 Matched Drawings

Another approach to relax requirements of SGE are the so-called matched drawings introduced by Di Giacomo et al. \[19\]. A matched drawing of a pair of graphs is a planar straight-line drawing of each of the graphs such that each common vertex has the same \( y \)-coordinate in both drawings (instead of the same \( y \)- and \( x \)-coordinate as required for SGE). Di Giacomo et al. \[19\] give a small counterexample consisting of two small triconnected planar graphs not admitting a matched drawing. Moreover, they give a larger example (620 vertices) of a biconnected graph and a tree not having a matched drawing. Apart from that they also have some results on the positive side. They show that two trees are always matched drawable. Moreover, they observe that any planar graph has a matched drawing with a so-called unlabeled level planar (ULP) graph, that is a graph that admits a planar straight-line drawing even if the \( y \)-coordinate of each vertex is prescribed such that no two vertices have the same \( y \)-coordinate. A characterization of ULP graphs is given by Fowler and Kobourov \[32\]. Di Giacomo et al. \[19\] moreover show for a graph class containing non-ULP graphs (the carousel graphs) that they admit matched drawings with
arbitrary planar graphs. A special cases of a carousel graphs is a graph consisting of a single vertex \( v_0 \) and a set of disjoint subgraphs \( S_1, \ldots, S_k \), each \( S_i \) connected to \( v_0 \) over a single edge \( \{v_0, v_i\} \) such that \( S_i \) is either a caterpillar with \( v_i \) on its spine, a radius-2 star with \( v_i \) as center or a cycle. Grilli et al. [39] present further positive results on matched drawings. They show how to draw the pairs outerplane plus wheel, wheel plus wheel, outerplane plus maximal outerpillar (outerplane graph with triangulated inner faces and caterpillar as weak dual), and outerplane plus generalized outerpath (outerpath where some edges on the outer face may be replaced by some small subgraphs). Moreover, they consider matched drawings for graph triples and give algorithms creating matched drawings of three cycles, and a caterpillar and two ULP graphs.

7 Other Simultaneous Representations

In a simultaneous drawing of planar graph and its dual each vertex in the dual graph is required to be placed inside the corresponding face of the primal graph. Moreover, no crossings are allowed except for crossings between a dual and its corresponding primal edge. Tutte [54] first considered this problem and showed that every triconnected planar graph admits a simultaneous straight-line drawing with its dual. However, the resulting drawings may have exponentially large area. Erten and Kobourov [25] provide a linear-time algorithm simultaneously embedding a triconnected planar graph and its dual on a grid of size \((2n-2) \times (2n-2)\) such that all edges are drawn as straight-line segments.

Brightwell and Scheinerman [12] show the existence of a simultaneous straight-line drawing of a triconnected planar graph and its weak dual such that the crossings between dual and the corresponding primal edges are right-angular crossings. Argyriou et al. [6, 5] give a simple example of a graph that is not triconnected not admitting such a drawing. On the positive side they give an algorithm that creates such drawings for the case that the primal graph is outerplanar.

Jampani and Lubiw [44] introduce the concept of simultaneous graph representations for other representations than drawings. An intersection representation of a graph assigns a geometric object to each vertex such that two vertices are adjacent if and only if their corresponding geometric objects intersect. Two graphs sharing a common subgraph are simultaneous intersection graphs if each of them has an intersection representation such that the common vertices are represented by the same objects. Note that every planar drawing of a graph can be interpreted as intersection representation, each vertex is represented by the union of its edges. This shows that deciding SEFE as a special case of recognizing simultaneous intersection graphs. Other popular intersection representations are the following. In an interval representation of a graph each vertex is represented by an interval on the real line. A graph is chordal if each induced cycle has length three. Gavril [36] shows that chordal graphs are exactly the intersection graphs of subtrees in a tree. This shows that the class of interval graphs is contained in the class of chordal graphs. Permutation graphs are the intersection graphs that can be represented by a set of line segments connecting two parallel lines. Jampani and Lubiw [44] give \(O(n^3)\)-time algorithms recognizing simultaneous permutation graphs and simultaneous chordal graphs. The algorithm for simultaneous permutation graphs can be extended to more than two graphs with sunflower intersection. On the other hand, it is NP-hard to recognize simultaneous chordal graphs of this kind (for a constant number \( k \) of graphs, the complexity is still open). In a follow-up paper Jampani and Lubiw [45] give an algorithm recognizing simultaneous interval graphs in \(O(n^2 \log n)\) time. As interval graphs can be characterized in terms of PQ-trees, recognizing simultaneous interval graphs leads to a problem of finding orders in several PQ-trees simultaneously. Bläsius and Rutter [8] consider this kind of problem in a more general leading to a \(O(n)\)-time algorithm recognizing simultaneous interval graphs.

Related to simultaneous intersection graphs are simultaneous comparability graphs also introduced by Jampani and Lubiw [44]. A comparability graph is a graph that can be oriented
transitively where transitively means that a directed path implies the existence of a directed edge. Two graphs are simultaneous comparability graphs if each of them can be oriented transitively such that common edges are oriented the same in both. Jampani and Lubiw give an \( O(nm) \)-time algorithm recognizing simultaneous comparability graphs. It can also be used to recognize an arbitrary number of comparability graphs with sunflower intersection. Comparability graphs are related to intersection graphs as comparability graphs are exactly the graphs whose complement is a function graph, that is the intersection graph with respect to continuous functions on an interval \([38]\).

As for the problem SEFE, finding simultaneous representations is related to extending a representation of a subgraph to one of the whole graph. For interval graphs Klavík et al. \([49]\) give a \( O(nm) \)-time algorithm testing whether a partial interval representation can be extended. Bläsius and Rutter \([8]\) were able to improve the running time to \( O(m) \) by constructing a second graph such that both graphs are simultaneous interval graphs if and only if the partial interval representation can be extended.

8 Practical Approaches

The majority of the results reviewed above focused on the theoretical aspects of the problem of embedding multiple graphs simultaneously. Numerous negative results show that in many of the interesting setting we cannot guarantee nice simultaneous embeddings. On the other hand, several efficient algorithms for different variants of the problem do exist, but they usually place additional restrictions on the number of input graphs, or limit the graphs to special sub-classes of planar graphs. As discussed in the introduction, the problem is well-motivated in practice. Of particular interest are applications to visualization of dynamic graphs and the related issues of mental map preservation and good graph readability. With this in mind we mention several more practical results here. For example, Erten et al. \([26]\) adapt force-directed algorithms to create drawings of a series of graphs sharing subgraphs finding a tradeoff between nice drawings and similarities of common parts. Kobourov and Pitta \([50]\) describe an interactive system which allows multiple users to interactively modify a pair of graphs simultaneously using a multi-user, touch-sensitive input device. While these two approaches focus on straight line drawings (corresponding to SGE), the GraphiSET system by Estrella-Balderrama et al. \([27]\) also allows edges to have bends. GraphiSET is a tool helping the user to investigate the theoretical problems SGE and SEFE and it contains implementations of several testing and drawing algorithms. Chimani et al. \([15]\) create simultaneous drawings of graphs by drawing the union of the graphs. Their objective is the number of crossings in the drawing, where crossings between edges of different graphs do not count, yielding a simultaneous embedding if and only if the number of crossings is zero.

9 Open Problems

There are many interesting problems, some of which have been open for a decade and have resisted efforts to address them. Here we list several of the current open problems.

1. Given two arbitrary planar graphs \( G^\emptyset = (V^\emptyset, E^\emptyset) \) and \( G^\emptyset = (V^\emptyset, E^\emptyset) \) with the same number of vertices, \( |V^\emptyset| = |V^\emptyset| \), does there always exist a mapping from the vertex set of the first graph onto the vertex set of the second graph \( V^\emptyset \to V^\emptyset \) such that the two graphs have a SGE? That is, do pairs of planar graphs always have an SGE without mapping?

2. Given two graphs of max-degree 2, \( G^\emptyset = (V^\emptyset, E^\emptyset) \) and \( G^\emptyset = (V^\emptyset, E^\emptyset) \) with the same number of vertices, an SGE with mapping does always exist. Unlike most other results where the pair of graphs has an SGE the area of the necessary grid is not bounded. Is it possible to guarantee polynomial integer grid for the simultaneous embedding?
3. What is the complexity of SGE for two graphs with fixed planar embeddings?

4. Is it possible to decide SGE for restricted cases, for example if the common graph is highly connected?

5. What is the complexity of the decision problem SEFE for two graphs?

6. Are there interesting parameters for which SEFE or SGE are FPT? For example, treedistance of $G$? What about maximum degree $\Delta$?

7. What is the complexity of SEFE for more than two graphs with sunflower intersection?

8. What is the complexity of SEFE for four graphs, each with a fixed planar embedding?

9. What is the complexity of the optimization version of SEFE where one asks for drawings such that as many common edges as possible are drawn the same?

10. Let $G^\circ$ and $G^\bullet$ be two planar graphs with given combinatorial embeddings inducing the same embedding on their intersection $G$, that is a SEFE is given with the input. What is the complexity of minimizing the number of crossings in a corresponding drawing?

11. Let $G^\circ$ and $G^\bullet$ be two planar graphs with given combinatorial embeddings inducing the same embedding on their intersection $G$, that is a SEFE is given with the input. Do $G^\circ$ and $G^\bullet$ admit drawings with few bends on a small grid respecting the given SEFE?

12. There are many open problems in the CSE setting. A particularly interesting one concerns pairs of trees. It is known that two $n$-vertex trees without mapping (1-colored) have a simultaneous geometric embedding (any set of $n$ points in convex position suffices). It is also known that at the other extreme when the mapping is given ($n$-colored) such geometric embedding may not exist. However, the problem is open for any number of colors $c \in \{2, \ldots, n-1\}$.

13. Similarly to the previous problem, the status of the tree-path CSE problem is open for any number of colors $c \in \{3, \ldots, n-1\}$.

References


