Tutte Embedding: How to Draw a Graph

Kyri Pavlou

Math 543 Fall 2008
Outline

• Problem definition & Background

• Barycentric coordinates & Definitions

• Tutte embedding motivation

• Barycentric Map Construction
  – Worked example
  – The linear system

• Drawbacks
Problem Definition

- **Graph Drawing:**
  Given a graph $G = (V, E)$ we seek an injective map (embedding)
  \[ f: V(G) \to \text{Space} \]
  such that $G$'s connectivity is preserved.

For this discussion:
- Space is $\mathbb{R}^2$.
- Edges are straight line segments.
Background

- Early graph drawing algorithms:

- These algorithms are force-directed methods. (a.k.a. spring embedders)
  - Vertices: steel rings
  - Edges: springs
  - Attractive/repulsive forces exist between vertices.
  - System reaches equilibrium at minimum energy.
Background: Tutte Embedding

- **William Thomas Tutte** (May 14, 1917 – May 2, 2002) was a British, later Canadian, mathematician and codebreaker.

- Tutte devised the first known algorithmic treatment (1963) for producing drawings for 3-connected planar graphs.

- Tutte constructed an embedding using barycentric mappings.

- The result is guaranteed to be a plane drawing of the graph.
Outline

• Problem definition & Background

• **Barycentric** coordinates & Definitions

• Tutte embedding motivation

• Barycentric Map Construction
  – Worked example
  – The linear system

• Drawbacks
Overview of barycentric coordinates

- Special kind of local coordinates
- Express location of point w.r.t. a given triangle.
- Developed by Möbius in the 19th century.
- Wachspress extended them to arbitrary convex polygons (1975).
- Introduced to computer graphics by Alfeld et al. (1996)
Why *barycentric*?

- \(v_G\) is the point where the medians are concurrent.

- \(v_G\) is called the barycenter or centroid and in physics it represents the center of mass.

- If \(v_G, v_1, v_2, v_3 \in \mathbb{R}^2\) then \(v_G\) can be easily calculated as:
  \[
  v_G = \frac{1}{3} \cdot (v_1 + v_2 + v_3)
  \]

- We want to extend this so that we can express every point \(v\) in terms of the vertices of a polygon \(v_1, v_2, \ldots, v_k\).
Convex Combinations

• If $P$ is a polygon with vertices $v_1, v_2, \ldots, v_k \in \mathbb{R}^2$ then we wish to find coordinates $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ such that for $v_0 \in ker(P)$

$$
\sum_{i=1}^{k} \lambda_i v_i = v_0
$$

• Note that if $\forall i \lambda_i > 0$ then $v_0$ lies inside the convex hull.
Useful definitions

- We say that a representation of $G$ is barycentric relative to a subset $J$ of $V(G)$ if for each $v$ not in $J$ the coordinates $f(v)$ constitute the barycenter of the images of the neighbors of $v$.

  where $f : V(G) \to \mathbb{R}^2$

- $k$-connected graph: If $G$ is connected and not a complete graph, its vertex connectivity $\kappa(G)$ is the size of the smallest separating set in $G$. We say that $G$ is $k$-connected if $\kappa(G) \geq k$.

  e.g. The minimum cardinality of the separating set of a 3-connected graph is 3.
Useful definitions\(^{(2)}\)

- Given \( H \leq_s G \), define relation \(~\) on \( E(G) - E(H) \):
  \( e \sim e_0 \) if \( \exists \) walk \( w \) starting with \( e \), ending with \( e_0 \), s.t. no internal vertex of \( w \) is in \( H \).

- **Bridge**: a subgraph \( B \) of \( G - E(H) \) if it is induced by \( ~ \).

- **A peripheral polygon**: A polygonal face \( P \) of \( G \) is called **peripheral** if \( P \) has at most 1 bridge in \( G \).
Outline

• Problem definition & Background

• Barycentric coordinates & Definitions

• Tutte embedding motivation

• Barycentric Map Construction
  – Worked example
  – The linear system

• Drawbacks
Tutte embedding motivation

• The idea is that if we can identify a peripheral $P$ then its bridge $B$ (if is exists) always avoids “all other bridges”… (True—there aren’t any others!)

• This means the bridge is transferable to the interior region and hence $P$ can act as the fixed external boundary of the drawing.

• All that remains then is the placement of the vertices in the interior.
Theorem: If $M$ is a planar mesh of a nodally 3-connected graph $G$ then each member of $M$ is peripheral.

In other words, Tutte proved that any face of a 3-connected planar graph is a peripheral polygon.

This implies that when creating the embedding we can pick any face and make it the outer face (convex hull) of the drawing.
Outline

• Problem definition & Background

• Barycentric coordinates & Definitions

• Tutte embedding motivation

• Barycentric Map Construction
  – Worked example
  – The linear system

• Drawbacks
Barycentric mapping construction

• Steps:

1. Let $J$ be a peripheral polygon of a 3-connected graph $G$ with no Kuratowski subgraphs ($K_{3,3}$ and $K_5$).
2. We denote the set of nodes of $G$ in $J$ by $V(J)$, and $|V(J)| = n$. Suppose there are at least 3 nodes of $G$ in the vertex set of $J$.
3. Let $Q$ be a geometrical $n$-sided convex polygon in Euclidean plane.
4. Let $f$ be a 1-1 mapping of $V(J)$ onto the set of vertices of $Q$ s.t. the cyclic order of nodes in $J$ agrees, under $f$, with the cyclic order of vertices of $Q$.
5. We write $m = |V(G)|$ and enumerate the vertices of $G$ as $v_1, v_2, v_3, \ldots, v_m$ so the first $n$ are the nodes of $G$ in $J$.
6. We extend $f$ to the other vertices of $G$ by the following rule. If $n < i \leq m$ let $N(i)$ be the set of all vertices of $G$ adjacent to $v_i$.
6. For each $v_i$ in $N(i)$ let a unit mass $m_j$ to be placed at the point $f(v_i)$. Then $f(v_i)$ is required to be the centroid of the masses $m_j$.

7. To investigate this requirement set up a system of Cartesian coordinates, denoting the coordinates of $f(v_i)$, $1 \leq i \leq m$, by $(v_{ix}, v_{iy})$.

8. Define a matrix $K(G) = \{C_{ij}\}$, $1 \leq (i,j) \leq m$, as follows.
   - If $i \neq j$ then $C_{ij} = -(\text{number of edges joining } v_i \text{ and } v_j)$
   - If $i = j$ then $C_{ij} = \text{deg}(v_i)$

9. Then the barycentric requirement specifies coordinates $v_{ix}$, $v_{iy}$ for $n < j \leq m$ as the solutions to the two linear systems

$$\sum_{j=1}^{m} C_{ij} v_{ix} = 0 \quad \sum_{j=1}^{m} C_{ij} v_{iy} = 0$$

where $n < i \leq m$. For $1 \leq j \leq n$ the coordinates are already known.
Consider the peripheral cycle $J$, $V(J) = \{v_J, v_2, v_3\}$

$G$ is 3-connected with unique cut set $\{v_2, v_3, v_4\}$
Example\textsuperscript{(2)}

- $V(J) = \{v_1, v_2, v_3\}$
- $N(4) = \{v_1, v_2, v_3, v_5\}$
- $N(5) = \{v_2, v_3, v_4\}$
- $K(G) = \begin{pmatrix}
3 & -1 & -1 & -1 & 0 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
0 & -1 & -1 & -1 & 3
\end{pmatrix}$
- Form the 2 linear systems for $i = 4, 5$. 

\[G:\] 
\[
\begin{align*}
&v_1 \\
&v_2 \\
&v_3 \\
&v_4 \\
&v_5
\end{align*}
\]
Example

- The linear systems

\[ C_{41} v_{1x} + C_{42} v_{2x} + C_{43} v_{3x} + C_{44} v_{4x} + C_{45} v_{5x} = 0 \rightarrow 4v_{4x} - 7 = v_{5x} \]
\[ C_{51} v_{1x} + C_{52} v_{2x} + C_{53} v_{3x} + C_{54} v_{4x} + C_{55} v_{5x} = 0 \rightarrow -v_{4x} + 3v_{5x} = 4 \]

\[ C_{41} v_{1y} + C_{42} v_{2y} + C_{43} v_{3y} + C_{44} v_{4y} + C_{45} v_{5y} = 0 \rightarrow 4v_{4y} - v_{5y} = 10 \]
\[ C_{51} v_{1y} + C_{52} v_{2y} + C_{53} v_{3y} + C_{54} v_{4y} + C_{55} v_{5y} = 0 \rightarrow -v_{4y} + 3v_{5y} = 4 \]

- Solutions

\[ v_4(25/11, 34/11) \quad v_5(23/11, 26/11) \]
Example: Tutte embedding

$\begin{align*}
\text{Example: Tutte embedding} \\
\text{v}_1&(3,6) \\
\text{v}_2&(0,3) \\
\text{v}_3&(4,1) \\
\text{v}_4&(25/11, 34/11) \\
\text{v}_5&(23/11, 26/11)
\end{align*}$
The linear system

• Is the linear system always consistent?
• Yes, it is!
• **Proof:**
  – Recall matrix $K(G)$.
    It was defined as $K(G) = \{ C_{ij} \}, \ 1 \leq (i,j) \leq m$.
    – If $i \neq j$ then $C_{ij} = -$(number of edges joining $v_i$ and $v_j$)
    – If $i = j$ then $C_{ij} = \deg(v_i)$
  – Observe that this means we can write $K(G)$ as
    $$K(G) = -A + D$$
    • where $A$ is the adjacency matrix of $G$ and
    • $D$ is diagonal matrix of vertex degrees.
  – But that’s the Laplacian of $G$! i.e., $K = -L$. 
The linear system (2)

- Let $K_1$ be the matrix obtained from $K(G)$ by striking out the first $n$ rows and columns.

  e.g. $K(G) = \begin{pmatrix}
    3 & 1 & 1 & -1 & 0 \\
    -1 & -1 & -1 & -1 & -1 \\
    -1 & 1 & 4 & 1 & 1 \\
    -1 & -1 & 4 & -1 & 0 \\
    0 & -1 & -1 & -1 & 3 
  \end{pmatrix}$

  $K_1 = \begin{pmatrix}
    4 & -1 \\
    -1 & 3 
  \end{pmatrix}$

- Let $G_0$ be the graph obtained from $G$ by contracting all the edges of $J$ while maintaining the degrees.

$G_0$: $v_{123} \quad v_4 \quad v_5$
The linear system \((3)\)

For a suitable enumeration of \(V(G_0)\), \(K_1\) is obtained from \(K(G_0)\) by striking out the first row and column.

\[-L(G_0) = K(G_0) = \begin{pmatrix}
5 & -3 & -2 \\
-3 & 4 & -1 \\
-2 & -1 & 3 \\
\end{pmatrix}\]

That is, \(K_1 = -\hat{L}_{11}\).

But then the \(\det(K_1) = \det(-\hat{L}_{11}) = t(G)\) is the number of spanning trees of \(G_0\).

\[\det(-\hat{L}_{11}) = \begin{vmatrix}
4 & -1 \\
-1 & 3 \\
\end{vmatrix} = 11\]
The linear system $(4)$

- The number $t(G)$ is non-zero since $G_0$ is connected.
  - Edge contraction preserves connectedness.

- This implies that $\det(K_1) \neq 0$ and the hence the linear systems always have a unique solution.
Outline

• Problem definition & Background

• Barycentric coordinates & Definitions

• Tutte embedding motivation

• Barycentric Map Construction
  – Worked example
  – The linear system

• Drawbacks
Drawbacks of Tutte Embedding

Only applies to 3-connected planar graphs.

Works only for small graphs ($|V| < 100$).

The resulting drawing is not always “aesthetically pleasing.”

Tutte representation: Dodecahedron

$Le(C60)$
References


Thank you!
Questions?