Maximum Differential Coloring of Caterpillars and Spiders

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Abstract. We study the maximum differential coloring problem, where the vertices of an n-vertex graph must be labeled with the numbers $1, ..., n$ such that the minimum difference between the two labels of any adjacent vertices is maximized. As it is NP-hard to find the optimal labels for general graphs, we consider special sub-classes: caterpillars, spiders, and extended stars. We first prove new upper bounds for maximum differential coloring for spiders and regular caterpillars. Using these new bounds, we prove that the Miller-Pritikin [18] labeling scheme for forests is optimal for regular caterpillars and for spider graphs. On the other hand, we give examples of general caterpillars where the Miller-Pritikin algorithm behaves very poorly. We present an alternative algorithm for general caterpillars which achieves reasonable results even in worst case. Finally we present new optimal labeling schemes for regular caterpillars and for sub-classes of spider graphs.

1 Introduction

The Four Color Theorem states that only four colors are needed to color any map so that no neighboring countries share the same color. However, if countries in the map are not all contiguous then the result no longer holds [7]. Instead, this necessitates the use of a unique color for each country to avoid ambiguity. As a result, the number of colors needed is equal to the number of countries.

Given a map, define the country graph $G = \{V, E\}$ to be the graph where countries are vertices and two countries are connected by an edge if they share a nontrivial boundary. In the *maximum differential coloring problem*, the goal is to find a labeling of the n vertices of G with the numbers 1 to n (treated as colors), which *maximizes* the label difference among adjacent vertices. More formally, let $C = \{c : V -> \{1, 2, ..., |V|\}$; c is one-to-one$\}$ be the set of one-to-one functions for labeling the vertices of G. For any $c \in C$ the *differential coloring* achieved by c is $\min_{(i,j)\in E} |c(i) - c(j)|$. We seek the labeling function $c \in C$ that achieves the maximum differential coloring for G, that is, function c which solves the Max-Min optimization problem:

$\mathrm{DC}(G) = \max_{c\in C} \min_{(i,j)\in E} |c(i) - c(j)|$.

$\mathrm{DC}(G)$ is known as the differential chromatic number of the graph G. The maximum differential coloring problem is the complement of a well-studied graph labeling problem known as *bandwidth minimization* [20]. There the objective is to find a one-to-one labeling of the vertices that *minimizes* the difference of labels between adjacent vertices. The problem remains NP-complete even for very restricted class of trees like caterpillars [19] and optimum solution is known only for very few classes of graphs like caterpillars with hair length at most 2 [1], chain graphs [14], cographs [26], bipartite permutation graphs [11] , AT-free graphs [8]. Although the maximum differential problem is less known than bandwidth minimization, it has received considerable attention recently. In addition to map-coloring, motivation can be found in the radio frequency assignment problem, where n transmitters have to be assigned n frequencies so that interfering transmitters have as different frequency as possible [10]. The problem also arises in the context of multiprocessor scheduling as well [16].

Previous Work. The maximum differential coloring problem was initially studied under the name "separation number" by Leung *et al.* [16] who showed the problem to be NP-complete. Yixun *et al.* [27] studies the same problem but under the name "dual-bandwidth" and gave several upper bounds, including the following simple bound for connected graphs:

Theorem 1. *[27] For any connected graph G,* $\mathrm{DC}(G) \leq \lfloor \frac{n}{2} \rfloor$.

The proof is straightforward: one of the vertices of G has to be labeled $\lceil \frac{n}{2} \rceil$ and since G is connected that vertex must have at least one neighbor which (regardless of its label) would make the difference along that edge at most $\lfloor \frac{n}{2} \rfloor$. Finally, this problem is also known as the "anti-bandwidth problem" [3].

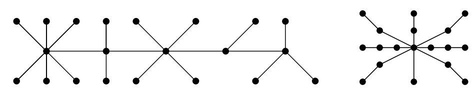
Heuristics for the maximum differential coloring problem have been suggested by Duarte *et al.* [5] using LP-formulation, by Bansal *et al.* [2] using memetic algorithms and by Hu *et al.* [12] using spectral based methods. The Miller-Pritikin [18] labeling scheme is defined for any forest G with disjoint vertex sets U and V and edges of the form $e = (u, v)$, where $u \in U$ and $v \in V$. The Miller-Pritikin scheme gives a differential coloring value equal to the size of the smaller vertex set i.e., $\min\{|U|, |V|\}$.

Another line of research focuses on finding the differential chromatic number for special classes of graphs, i.e., solving the maximum differential coloring problem optimally on special graphs. The differential chromatic number is known for Hamming graphs [4],meshes [21], hypercubes [21,24], complete binary tree [25] and complete k-ary trees for odd values of k [3]. Isaak *et al.* [13] gives a greedy algorithm to compute the differential chromatic number of complement of interval graphs, threshold graphs and arborescent comparability graphs by computing the k-th power of a Hamiltonian path. Also tight upper bounds are known for 3D meshes [23] and d-D meshes [22]. Weili *et al.* [25] also compute the differential chromatic number for special graphs which they call caterpillars (however, their caterpillars are not the standard caterpillar graphs we are interested in).

Equitable coloring [9] is an assignment of colors to an unweighted graph so that no two adjacent vertices have the same color and the number of vertices in any two graph classes differ by at most one. The problem of deciding if a graph has equitable coloring ≤ 3 was proved NP-complete by Kubale [15] and asymptotic bounds are known for some special graphs like k-uniform hypergraphs [28]. If the graph G has a differential chromatic number k, then the vertices labeled $[1, k], [k + 1, 2k] \cdots$ forms equitably colored classes and so G has an equitable coloring of $\lfloor \frac{n}{k} \rfloor$ [17]. Lin *et al.*-[17] gives a sub-optimal labeling for general graphs by using the relationship between the antibandwidth problem and the equitable coloring problem. They also provide a sub-optimal labeling for connected bipartite graphs with a differential coloring $\lfloor \frac{n}{\Delta} \rfloor$, where Δ is the max degree.

Our Contributions: We study the maximum differential coloring of subclasses of trees and in particular caterpillars, spiders, and extended stars, which naturally arise in level planarity, where caterpillars, degree-3 spiders, and radius-2 stars completely characterize the class of level planar trees [6]. Our main contributions are (i) A tight upper bound for regular caterpillars and spiders (Theorem 2,3). (ii) A labeling which produces a close to optimal labeling for caterpillars (Theorem 6). (iii) A closed form optimal labeling scheme (more intuitive than the already existing optimal labeling scheme [18]) for regular caterpillars and for spiders with all even or all odd length paths (Theorem 4,5).

A *caterpillar* is a tree in which removing all leaves results in a path. Thus, a caterpillar consist of a simple path, called the "spine", and each spine vertex is adjacent to a certain number of leaves, called the "legs". In caterpillars, Δ refers to the maximum number of legs of any spine vertex. In a *regular caterpillar* every spine vertex has the same number of legs. A *spider* is a graph with a center vertex v_c connected to p paths. The vertices of a spider have *levels*, according to their distance from center v_c. See Figure 1.



(a) Caterpillar. (b) Spider.

Fig. 1: A caterpillar and spider.

We provide tight upper bounds for spiders and regular caterpillar graphs. For these graphs the best known previous upper bound result was the trivial $\lfloor \frac{n}{2} \rfloor$. We prove the following.

Theorem 2. *Let G be a regular caterpillar with n vertices. If G has an odd number of spine vertices then* $\mathrm{DC}(G) \leq \lceil \frac{n-\Delta}{2} \rceil$. *Otherwise* $\mathrm{DC}(G) \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 3. *If G is a spider graph with N_e even level vertices, then* $\mathrm{DC}(G) \leq N_e + 1$.

Using Theorem 2 and Theorem 3 we prove that the Miller-Pritikin [18] labeling scheme for forests is optimal for regular caterpillars and for spider graphs.

Corollary 1. *The Miller-Pritikin labeling scheme is optimal for regular caterpillars.*

Corollary 2. *The Miller-Pritikin labeling scheme is optimal for spiders.*

In Section 3 and 4 we present alternative optimal labeling schemes for regular caterpillars and for spiders with all even or all odd length paths. Even though our labeling scheme does not constitute any improvement over the already existing labeling schemes [18] our labeling scheme is significantly more intuitive. We prove the following.

Theorem 4. *Let G be a regular caterpillar with n vertices. There exists an optimal labeling of G with value $\lfloor \frac{n}{2} \rfloor$ when G has an even number of spine vertices, and with value $\lceil \frac{n-\Delta}{2} \rceil$ otherwise.*

Theorem 5. *Let G be a spider graph with N_e even level vertices. If the paths of G are all of odd length or all of even length, there exist optimal labelings for G with value $N_e + 1$ and N_e, respectively.*

A *radius-k star* graph is a spider with all paths of the same length k. As corollary of Theorem 5 in Section 4 we show the following.

Corollary 3. *There exists an optimal labeling for all radius-k star graphs.*

Finally we design a new labeling scheme for general caterpillars. The Miller-Pritikin labeling [18] can also be used for general caterpillars but it is by far not optimal. In Section 5.3 we show that it can be worse by a factor of $\frac{\Delta}{4}$ than our labeling using Theorem 6. We prove the following.

Theorem 6. *Let G be a caterpillar with n vertices. There exists a labeling of G with differential coloring value at least $\lceil \frac{n}{2} \rceil - \Delta - 2$.*

2 Upper Bounds for Regular Caterpillars and Spiders

We establish new upper bounds for $\mathrm{DC}(G)$ when G is a regular caterpillar or a spider proving Theorems 2 and 3. Using these theorems we show that the Miller-Pritikin labeling is optimal for these classes of graphs, as stated by Corollaries 1 and 2.

Theorem 2. *Let G be a regular caterpillar with n vertices. If G has an odd number of spine vertices then $\mathrm{DC}(G) \leq \lceil \frac{n-\Delta}{2} \rceil$. Otherwise $\mathrm{DC}(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Proof: Let G be a regular caterpillar. If G has an even number of spine vertices $\mathrm{DC}(G) \leq \lfloor \frac{n}{2} \rfloor$ by Theorem 1. We will show that, when G has an odd number of spine vertices, $\mathrm{DC}(G) \leq \lceil \frac{n-\Delta}{2} \rceil$. Let the number of spine vertices be $s = 2k + 1$ so that $n = (2k + 1)(\Delta + 1)$. For a proof by contradiction assume there exists a labeling of value $c^* = \lceil \frac{n-\Delta}{2} \rceil + 1$.

Lemma 1. *No spine vertex is labeled with a number in the interval $[\lceil \frac{n-\Delta}{2} \rceil, \lceil \frac{n+\Delta}{2} \rceil]$*

Proof: Suppose $i \in [\lceil \frac{n-\Delta}{2} \rceil, \lceil \frac{n+\Delta}{2} \rceil]$ is a spine vertex label. Consider the labels that can be assigned to the Δ legs of i (with a slight abuse of notation i also refers to the vertex labeled i). To achieve value c^* the label for a leg of i can either lie in the interval $L = [1, i - (\lceil \frac{n-\Delta}{2} \rceil + 1)]$ or in the interval $H = [i + \lceil \frac{n-\Delta}{2} \rceil + 1, n]$. Consider three cases for the label of i. Noting that $\lceil \frac{n-\Delta}{2} \rceil = 1 + k(\Delta + 1)$,
Case 1: $i \neq \lceil \frac{n-\Delta}{2} \rceil$ and $i \neq \lceil \frac{n+\Delta}{2} \rceil$. The total number of labels in L and H is

$$\begin{aligned} i - \left(\left\lceil \frac{n-\Delta}{2} \right\rceil + 1 \right) + n - \left(i + \left\lceil \frac{n-\Delta}{2} \right\rceil \right) &= n - \left(\left\lceil \frac{n-\Delta}{2} \right\rceil + 1 + \left\lceil \frac{n-\Delta}{2} \right\rceil \right) \\ &= n - (2k(\Delta + 1) + 3) \\ &= n - (n + 2 - \Delta) \ = \ \Delta - 2 \end{aligned}$$

Case 2: $i = \lceil \frac{n-\Delta}{2} \rceil$. L is empty and all leg labels lie in interval H. The total number of labels in H is

$$n - \left(2 \cdot \left\lceil \frac{n-\Delta}{2} \right\rceil\right) = n - (2 \cdot ((k \cdot (\Delta + 1)) + 1)$$
$$= ((2k+1) \cdot (\Delta + 1)) - (2 \cdot ((k \cdot (\Delta + 1)) + 1)) \;\; = \;\; \Delta - 1$$

Case 3: $i = \lceil \frac{n+\Delta}{2} \rceil$. H is empty and all leg labels lie in interval L. The total number of labels in L is

$$i - \left(\left\lceil \frac{n-\Delta}{2} \right\rceil + 1\right) = \Delta - 1$$

In all cases the labels for the legs of i are insufficient, proving the lemma. □

By Lemma 1 labels of spine vertices either lie in interval $L_s = [1, \lceil \frac{n-\Delta}{2} \rceil - 1]$ or $H_s = [\lceil \frac{n+\Delta}{2} \rceil + 1, n]$. Observe that the maximum difference between any two elements in the interval L_s is $\lceil \frac{n-\Delta}{2} \rceil - 2$. Thus to achieve differential coloring c^*, adjacent spine vertices cannot both be labeled from the interval L_s. Similarly the maximum difference between two elements in H_s is $n - (\lceil \frac{n+\Delta}{2} \rceil + 1) = n - (\lceil \frac{n-\Delta}{2} \rceil + \Delta + 1) = n - (k \cdot (\Delta + 1) + 1 + \Delta + 1) = \lceil \frac{n-\Delta}{2} \rceil < c^*$, adjacent spine vertices cannot both be labeled from the interval H_s. Thus the labels for spine vertices must alternate between interval L_s and H_s such that for the labels of the $(2k+1)$ spine vertices, one of the intervals supplies $(k+1)$ labels and other interval supplies k labels. Assume w.l.o.g that L_s supplies $(k+1)$ labels. To achieve c^* the $(k+1)\Delta$ legs of these spine vertices must all have labels in the interval $I = [\lceil \frac{n-\Delta}{2} \rceil + 2, n]$. As $\Delta \geq 1$, interval $I \supset H_s$, and so I must also contain the k labels H_s supplies for spine vertices. Thus in total I must contain at least $(k+1)\Delta + k$ labels. However the size of the interval I is only $1+n-(\lceil \frac{n-\Delta}{2} \rceil+2) = 1+((2k+1)\cdot(\Delta+1))-((k\cdot(\Delta+1))+1)-2 = k\cdot\Delta+k+\Delta-1$, so we have a contradiction. □

Corollary 1. *The Miller-Pritikin labeling scheme is optimal for regular caterpillars.*

Proof: A regular caterpillar G with even number of spine vertices $s = 2k$ is a bipartite graph whose vertices form disjoint sets U and V, where U consists of the k odd spine vertices and the $k\Delta$ legs of the even spine vertices, and V consists of the k even spine vertices and the $k\Delta$ legs of the odd spine vertices. So $|U| = |V| = k + k\Delta = \frac{n}{2}$. Since the Miller-Pritikin scheme gives a labeling with value equal to the size of the smaller vertex set, the labeling is optimal. Now consider G with odd number of spine vertices $s = 2k+1$. U consists of the k even spine and the $(k+1)\Delta$ legs of the odd spine vertices, and V consists of $k+1$ odd spine vertices and the $k\Delta$ legs of the even spine vertices. So $|U| = k+(k+1)\Delta$ and $|V| = k+1+k\Delta$, and $\min\{|U|, |V|\} = k+1+k\Delta = \lceil \frac{n-\Delta}{2} \rceil$. Thus by Theorem 2 the Miller-Pritikin labeling scheme is optimal. □

Theorem 3. *If G is a spider graph with N_e even level vertices, then* $\mathrm{DC}(G) \leq N_e + 1$.

Proof: We make a few simple observations about spider graphs before presenting the proof of Theorem 3. Let p be the number of paths connected to the center vertex v_c in a spider graph G and let N_o, N_e and N_i denote the number of even-level, number of odd-level and number of vertices in level i respectively. The number of vertices in G is

$$n = N_e + N_o + 1 \tag{1}$$

Each of the p paths in G starts with an odd-level vertex and alternates between even and odd levels. It follows that on each path the number of odd level vertices is at most one more than the even level vertices. Summing over all p paths we get

$$N_o - N_e \leq p \tag{2}$$

Now we prove that for a spider G with N_e even-level vertices $\mathrm{DC}(G) \leq N_e + 1$. For a proof by contradiction suppose there exists a labeling of value $c^* = N_e + 2$.

Lemma 2. *The center vertex label is not in the interval* $[N_e + 1, N_e + p + 1]$.

Proof: For the sake of contradiction, let $i \in [N_e + 1, N_e + p + 1]$ be the label of the center vertex and consider the labels that can be assigned to the p vertices of level 1. To achieve a differential coloring of value c^*, the labels of the level 1 vertices can either lie in the interval $L = [1, i - (N_e + 2)]$ or in the interval $H = [N_e + 2 + i, n]$. Consider three cases for values of i.

Case 1: $i \neq N_e + 1$ and $i \neq N_e + p + 1$. Then, the total number of labels in L and H is

$$\begin{aligned} i - N_e - 2 + n - (i + N_e + 2) + 1 &= i - N_e - 2 + N_e + N_o + 1 - (i + N_e + 2) + 1 \\ &= N_o - N_e - 2 \leq p - 2, \end{aligned}$$

by Equations 1 and 2.

Case 2: $i = N_e + 1$. L is empty and all labels of level 1 vertices lie in H. The total number of labels in H is

$$\begin{aligned} n - (i + N_e + 2) + 1 &= n - (N_e + 1 + N_e + 2) + 1 \\ &= N_e + N_o + 1 - (N_e + 1 + N_e + 2) + 1 \\ &= N_o - N_e - 1 \leq p - 1 \quad , \end{aligned}$$

by Equations 1 and 2.

Case 3: $i = N_e + p + 1$. H is empty and all labels of level 1 vertices lie in L. The total number of labels in L is

$$i - N_e - 2 = N_e + p + 1 - N_e - 2 = p - 1$$

In all cases the number of labels for level 1 vertices is less than p, proving the lemma. □

By Lemma 2 the center label either lies in interval $L_c = [1, N_e]$ or in interval $H_c = [N_e + p + 2, n]$. Let us first assume that the center label lies in L_c . The level 1 vertices should then lie in the interval $I = [N_e + 3, n]$. Note that to achieve a maximum difference value of $c^* = N_e + 2$, adjacent vertices from neighboring levels $2j$ and $2j+1$ cannot both lie in the interval $[1, N_e+2]$. Also, $N_{2j+1} \leq N_{2j}$. Thus the labels of at least N_{2j+1} vertices lie in the interval I. So interval I must contain at least $N_1 + N_3 + \cdots = N_o$ elements. However the size of the interval I is $n - N_e - 3 + 1 = N_e + N_o + 1 -$

$N_e - 3 + 1 = N_o - 1$, a contradiction. Now assume that center lies in the interval H_c. An analogous argument shows that the interval $I' = [1, n - N_e - 2]$ must contain at least N_o elements which is more than the size of I', leading to a contradiction. As both cases result in contradictions, this completes the proof of Theorem 3. □

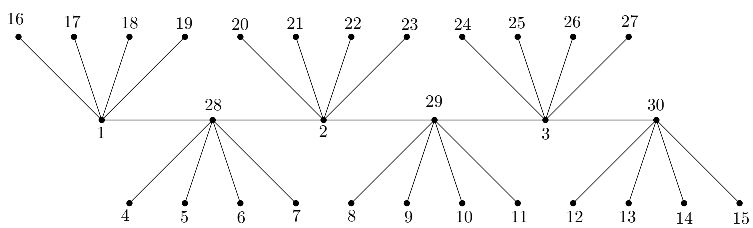
Corollary 2. *The Miller-Pritikin labeling scheme is optimal for spiders.*

Proof: A spider graph G is a bipartite graph whose vertices form disjoint sets U and V, where the even level vertices and the center vertex form U and the odd level vertices form V. Labeling G with the Miller-Pritikin [18] scheme gives a differential coloring of value at least $m = \min\{|U|, |V|\} = \min\{N_e + 1, N_o\}$.

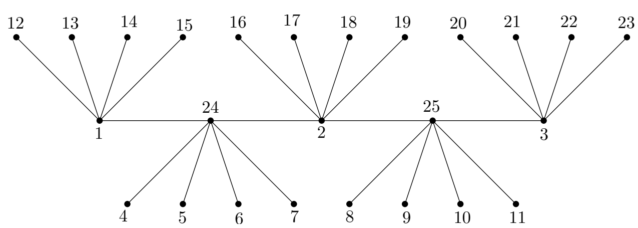
We now prove that m is optimal. We have that $N_e \leq N_o$. If $N_e = N_o$, then $m = N_o$. By Theorem 1 DC(G) is at most $\lfloor n/2 \rfloor$ which by Equation 1 is at most $\lfloor (N_o + N_e + 1)/2 \rfloor = \lfloor (2N_o + 1)/2 \rfloor = N_o = m$. Now assume $N_e < N_o$. Then $m = N_e + 1$ which is optimal by Theorem 3, completing the proof. □

3 Optimal labeling for regular caterpillars

Let G be an n-vertex regular caterpillar with each spine vertex having $\Delta \geq 1$ legs. Let s denote the number of spine vertices. Then as $n = s \cdot (\Delta + 1)$, we have $s = \frac{n}{\Delta+1}$.



(a) Even number of spine vertices.



(b) Odd number of spine vertices.

Fig. 2: Our optimal labeling for regular caterpillars.

Observation: It is always good to assign an interval of consecutive numbers to all the legs of a spine vertex v since the max difference between a spine vertex v and its legs depends only on the difference between the label of v and the highest or lowest label of the legs of v.

First consider the case that there is an even number of spine vertices, that is, $s = 2k$. We assign the k lowest and k highest numbers in an alternating fashion to the spine vertices. Starting with the leftmost spine vertex and moving to the right we label the spine vertices as $1, n-k+1, 2, n-k+2, \ldots$, and so on, ending at the rightmost spine vertices with numbers k and n. See Figure 2a. There are only two values for the differences between adjacent spine vertices, namely $n-k$ and $n-k-1$, and as $\Delta \geq 1$, the difference is at least $n-k-1 = 2k\Delta + k - 1 \geq k(\Delta+1) = n/2$. We denote L_s to be the set of spine vertices with labels from $1, \ldots, k$ and H_s as the spine vertices with labels from $n-k+1, \ldots n$.

Next we split the middle range $[k+1, n-k]$ into two ranges $L_\ell = [k+1, \frac{n}{2}]$ and $H_\ell = [\frac{n}{2}+1, n-k]$. We assign labels to the legs of L_s from range H_ℓ and to the legs of H_s from L_ℓ as follows. For a spine vertex from L_s with label $1 \leq i \leq k$ we assign its Δ legs to the numbers from interval $[\frac{n}{2} + ((i-1)\cdot\Delta) + 1, \frac{n}{2} + i\cdot\Delta]$. For a spine vertex from H_s with label $j = n-k+i$ between $n-k+1$ and n, we assign the numbers from the interval $[k + ((i-1)\cdot\Delta) + 1, k + i\cdot\Delta]$ to its Δ legs.

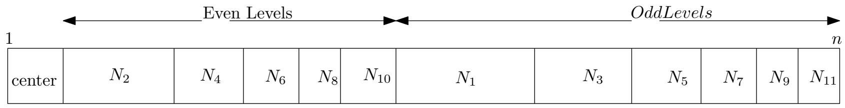
Thus the difference between a low spine vertex from L_s and one of its legs is at least $\frac{n}{2} + (i-1)\cdot\Delta + 1 - i = n/2 + i(\Delta-1) - \Delta + 1$ which is minimum for $i = 1$, namely $\frac{n}{2}$. Analogously, the difference between a high spine vertex $j = n-k+i$ and one of its legs is at least $j - (k + i\cdot\Delta) = n-k+i-k-i\cdot\Delta = n-2k+i(1-\Delta)$ which is minimal for largest possible i. Thus setting $i = k$, and using the fact that $k = \frac{n}{2}(\Delta+1)$, the difference is again $\frac{n}{2}$. Hence there exists a labeling for which the max difference is $\frac{n}{2}$.

Now consider the case when the number s of spine vertices is odd, say $s = 2k+1$. See Figure 2b. We follow the scheme as above assigning the lowest $k+1$ numbers and highest numbers k in an alternating fashion to the spine vertices. The differences between adjacent spine vertices are $n-k$ and $n-k-1$ which is at least $n-k-1 = 2k\Delta + \Delta + k \geq k\Delta + 1 + k \geq \lceil \frac{n-\Delta}{2} \rceil$, as $\Delta \geq 1$.

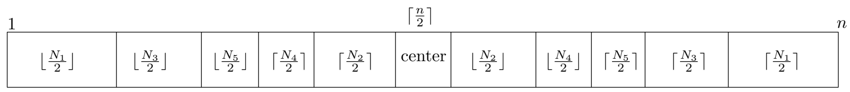
Let L_s denote the spine vertices with labels $\leq k+1$ and H_s be the spine vertices with labels $> n-k$. As before we divide the middle range $[k+2, n-k]$ into two ranges $L_\ell = [k+2, \lceil \frac{n-\Delta}{2} \rceil]$ and $H_\ell = [\lceil \frac{n-\Delta}{2} \rceil + 1, n-k]$ and assign the numbers from L_ℓ to the legs of H_s and the numbers from H_ℓ to the legs of L_s. For a spine vertex from L_s with label $1 \leq i \leq k+1$, we assign the numbers from the interval $[\lceil \frac{n-\Delta}{2} \rceil + ((i-1)\cdot\Delta) + 1, \lceil \frac{n-\Delta}{2} \rceil + i\cdot\Delta]$ to its Δ legs. For a spine vertex from H_s with label $j = n-k+i$ between $n-k+1$ to n, we assign numbers to its Δ legs from the interval $[k + ((i-1)\cdot\Delta) + 2, k + i\cdot\Delta + 1]$.

The difference between a low spine vertex i and its legs is at least $\lceil \frac{n-\Delta}{2} \rceil + ((i-1)\cdot\Delta) + 1 - i$ which is minimal for $i = 1$ namely $\lceil \frac{n-\Delta}{2} \rceil$ as above. The difference between a high number $j = n-k+i$ and its legs is at least $j - (k + i\cdot\Delta + 1) = n-k+i-(k+i\cdot\Delta+1) = n-2k+i(1-\Delta)-1$ which is minimal for i as large as possible. Thus setting $i = k$ the difference is $n-2k+k(1-\Delta)-1 = n-k-k\Delta-1 = n-k(1+\Delta)-1$. As $k = \frac{n-(\Delta+1)}{2(\Delta+1)}$ we have that the difference is $\frac{n+\Delta}{2} - 1/2$, which is at least $\lceil \frac{n-\Delta}{2} \rceil$ as $\Delta \geq 1$. This gives the bound stated in Theorem 4.

Optimality. For a regular caterpillar G with even number of spine vertices our labeling achieves difference $\frac{n}{2}$ and for odd number of spine vertices it achieves difference $\lceil \frac{n-\Delta}{2} \rceil$. Both of these are optimal by Theorem 2.



(a) Case 1: For spiders with all even length path.



(b) Case 2: For spiders with all odd length paths.

Fig. 3: The two cases of our spider labeling scheme.

4 Labeling spiders

In this section we prove Theorem 5. Let G be a n-vertex spider with p paths. We consider the cases where all p paths are either of odd length or of even length separately. Let N_l denote the number of vertices at level l. For each $q \in [1, N_l]$ let $v_{l,q}$ be the level l vertex that belongs to the q-th path out of the paths containing level l vertices.

Case 1: All p paths have even length. See Figure 3a.

We label the center vertex as 1. The N_e even level vertices will be assigned values from interval $I_e = [2, N_e + 1]$ in increasing order of levels i.e. starting with the level 2 vertices, followed by level 4 vertices, etc. The N_o odd level vertices will be assigned values from interval $I_o = [N_e+2, n]$ in the same way i.e. with level 1 vertices, followed by the level 3 vertices, etc. For each level we will order the vertices in decreasing order of the lengths of the paths they belong to. More specifically we initially order the p paths in decreasing order of their lengths. For vertex $v_{l,q}$ belonging to the odd level $l = 2i + 1$ $i \geq 1$, we assign $N_e + 1 + \sum_{k=0}^{i-1} N_{2k+1} + q$. For vertex $v_{l,q}$ belonging to the even level $l = 2i$ $i \geq 1$ we assign $1 + \sum_{k=1}^{i-1} N_{2k} + q$.

We now show that the above labeling has maximum differential value N_e. First consider the difference between the center and a level 1 vertex $v_{1,q}$. The difference is $N_e + 1 + q - 1$. As $q \in [1, p]$, this is at least $N_e + 1 \geq N_e$.

Now consider the difference between a vertex of level $l = 2i$ and a vertex of level $l + 1 = 2i + 1$ with $i \geq 1$. $v_{l+1,q} - v_{l,q} = N_e + 1 + \sum_{k=0}^{i-1} N_{2k+1} + q - (1 + \sum_{k=1}^{i-1} N_{2k} + q) = N_e + N_{2(i-1)+1} + \sum_{k=0}^{i-2}(N_{2k+1} - N_{2k+2})$. Since $N_{2k+1} \geq N_{2k+2}$, the difference is at least $N_e + N_{2(i-1)+1} \geq N_e$.

Now consider the difference between a vertex $v_{l,q}$ at level $l = 2i$ and a vertex $v_{l-1,q}$ on the same path. Then $v_{l-1,q} - v_{l,q} = N_e + 1 + \sum_{k=0}^{i-2} N_{2k+1} + q - (1 + \sum_{k=1}^{i-1} N_{2k} + q) = N_e + \sum_{k=0}^{i-2}(N_{2k+1} - N_{2k+2})$. Since $N_{2k+1} \geq N_{2k+2}$ the difference is at least N_e. Thus the difference is at least N_e. Since all the paths have even length $N_e = N_o = \lceil \frac{n}{2} \rceil$.

Case 2: All of the p paths have odd length.See Figure 3b.

Let k be the length of the longest path. Since all the paths have odd length we have $N_{2i} = N_{2i+1} \forall i \in [1, \lfloor \frac{k}{2} \rfloor]$. Also $N_o - N_e = N_1 + \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor}(N_{2i+1} - N_{2i}) = p$. Also $n = N_o + N_e + 1 = 2N_e + p + 1$. So n and p have different parity. Also $\lceil \frac{n-p}{2} \rceil = N_e + 1$.

We label the center vertex as $\lceil \frac{n}{2} \rceil$. Order the paths as $P_{\lfloor \frac{p}{2}+1 \rfloor}, P_1, P_{\lfloor \frac{p}{2}+2 \rfloor}, P_2 \dots$ in the decreasing order of their lengths.

For vertex $v_{l,q}$ belonging to the odd level $l = 2i - 1$ and $q \leq \lfloor \frac{p}{2} \rfloor$, we assign $\sum_{m=1}^{i-1} \lfloor \frac{N_{2m-1}}{2} \rfloor + q$. And for vertex $v_{l,q'}$ belonging to the odd level $l = 2i - 1$ and $q' > \lfloor \frac{p}{2} \rfloor$, $q' = q + \lfloor \frac{p}{2} \rfloor$ we assign $n - (\sum_{m=1}^{i} \lceil \frac{N_{2m-1}}{2} \rceil) + q$.

For vertex $v_{l,q}$ belonging to the even level $l = 2i$ and $q \leq \lfloor \frac{p}{2} \rfloor$, we assign $\lceil \frac{n}{2} \rceil + \sum_{m=1}^{i-1} \lfloor \frac{N_{2m}}{2} \rfloor + q$. And for vertex $v_{l,q'}$ belonging to the even level $l = 2i$ and $q' > \lfloor \frac{p}{2} \rfloor$, $q' = q + \lfloor \frac{p}{2} \rfloor$ we assign $\lceil \frac{n}{2} \rceil - (\sum_{m=1}^{i} \lceil \frac{N_{2m}}{2} \rceil) + q - 1$.

We now show that the above labeling has maximum differential value $N_e + 1$. First consider the difference between the center and a level 1 vertex $v_{1,q}$. When $q \leq \lfloor \frac{p}{2} \rfloor$ the difference is $\lceil \frac{n}{2} \rceil - q$ which is at least $\lceil \frac{n}{2} \rceil - \lfloor \frac{p}{2} \rfloor = \lceil \frac{n-p}{2} \rceil$. When $q' > \lfloor \frac{p}{2} \rfloor, q' = q + \lfloor \frac{p}{2} \rfloor$ the difference is $n - \lceil \frac{N_1}{2} \rceil + q - \lceil \frac{n}{2} \rceil$ which is at least $n - \lceil \frac{p}{2} \rceil + 1 - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor - \lceil \frac{p}{2} \rceil + 1 = \lceil \frac{n-p}{2} \rceil$.

Now consider the difference between the labels of vertices in level $l = 2i$ and level $l - 1 = 2i - 1$. For $q \leq \lfloor \frac{p}{2} \rfloor$ the difference is $v_{l-1,q} - v_{l,q} = \lceil \frac{n}{2} \rceil + \sum_{m=1}^{i-1} \lfloor \frac{N_{2m}}{2} \rfloor + q - (\sum_{m=1}^{i-1} \lfloor \frac{N_{2m-1}}{2} \rfloor) + q) \geq \lceil \frac{n}{2} \rceil$. Because $\lfloor \frac{N_{2m}}{2} \rfloor \geq \lfloor \frac{N_{2m-1}}{2} \rfloor$. For $q' = q + \lfloor \frac{p}{2} \rfloor$ and $q' > \lfloor \frac{p}{2} \rfloor$ the difference is $v_{l-1,q'} - v_{l,q'} = n - (\sum_{m=1}^{i} \lceil \frac{N_{2m-1}}{2} \rceil) + q - (\lceil \frac{n}{2} \rceil - (\sum_{m=1}^{i} \lceil \frac{N_{2k}}{2} \rceil) + q - 1) \geq \lfloor \frac{n}{2} \rfloor + 1$. Because $\lfloor \frac{l_{2m}}{2} \rfloor \geq \lfloor \frac{l_{2m-1}}{2} \rfloor$. Also $\lfloor \frac{n}{2} \rfloor + 1 \geq \lceil \frac{n}{2} \rceil$. So differential is $\geq \lceil \frac{n}{2} \rceil$ in both cases.

Now consider the difference between a vertex $v_{l,q}$ at level $l = 2i$ and a vertex $v_{l+1,q}$ on the same path. For $q \leq \lfloor \frac{p}{2} \rfloor$ the difference is $v_{l+1,q} - v_{l,q} = \lceil \frac{n}{2} \rceil + \sum_{m=1}^{i-1} \lfloor \frac{N_{2m}}{2} \rfloor + q - (\sum_{m=1}^{i} \lfloor \frac{N_{2m-1}}{2} \rfloor) + q) \geq \lceil \frac{n}{2} \rceil - \lfloor \frac{N_{2i-1}}{2} \rfloor \geq \lceil \frac{n}{2} \rceil - \lfloor \frac{p}{2} \rfloor = \lceil \frac{n-p}{2} \rceil$. For $q' = q + \lfloor \frac{p}{2} \rfloor$ and $q' > \lfloor \frac{p}{2} \rfloor$ the difference is $v_{l-1,q'} - v_{l,q'} = (n - (\sum_{m=1}^{i+1} \lceil \frac{N_{2m-1}}{2} \rceil) + q) - (\lceil \frac{n}{2} \rceil - (\sum_{m=1}^{i} \lceil \frac{N_{2m}}{2} \rceil) + q - 1) = \lfloor \frac{n}{2} \rfloor + 1 - \lceil \frac{N_{2i+1}}{2} \rceil \geq \lfloor \frac{n}{2} \rfloor + 1 - \lceil \frac{p}{2} \rceil = \lceil \frac{n-p}{2} \rceil$

We now argue that we achieve an optimal labeling for G if all paths are of even length. This is case 1 and as shown above our labeling achieves $\mathrm{DC}(G) = N_e$. As all paths have even lengths $N_e = N_o$. Thus $n = N_e + N_o + 1 = 2N_e + 1$. By Theorem 1 $\mathrm{DC}(G) \leq \lceil n/2 \rceil = \lceil (2N_e + 1) \rceil = N_e$. See Figure 4a.

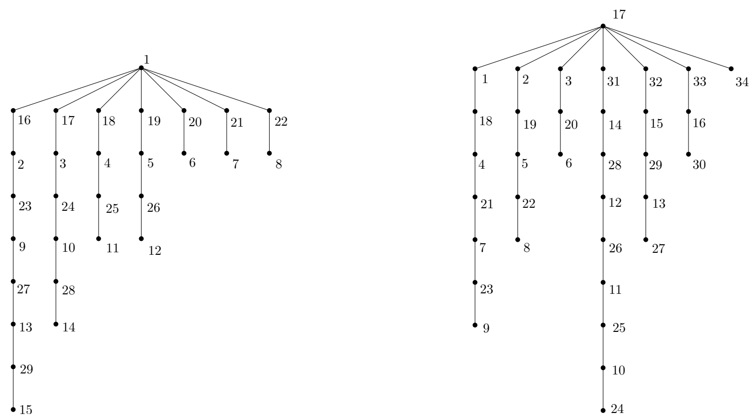
When G is a spider with all paths of odd length, we are in case 2 where our labeling achieves a maximum differential value of $N_e + 1$. This is optimal by Theorem 3. See Figure 4b.

4.1 Proof of Corollary 3

A k-radius star G is a spider where all paths have length exactly k. As k is either an even or an odd number either all paths of G are of even length or all paths are of odd length. Thus the labeling of Theorem 5 is optimal for G.

5 Labeling Caterpillars

We start with a labeling for the more intuitive but slightly restricted case where G is a caterpillar and each spine vertex has at least one leg. Then we adapt the labeling to general caterpillars.



(a) All paths of even lengths. (b) All paths of odd lengths.

Fig. 4: Special cases of spider graphs where our labeling is optimal.

5.1 Labeling for caterpillars with no missing legs

We describe a labeling algorithm for G that achieves differential value at least $n/2-\Delta-1$, comprising of two phases, the *marking phase* and the *labeling phase*. The marking phase places the vertices of G into one of the following sets: L_s, H_s, M, L_ℓ, H_ℓ, L_{M_ℓ} and H_{M_ℓ}. Vertices in L_s, H_s, M are spine vertices and those in L_ℓ, H_ℓ, $\{L_{M_\ell} \cup H_{M_\ell}\}$ are the legs of spine vertices in H_s, L_s and M respectively. The labeling phase assigns actual values to vertices in G. More precisely, the vertex in M is assigned value $\lceil n/2 \rceil$, the vertices in L_s, L_ℓ, L_{M_ℓ} are assigned low values $< \lceil n/2 \rceil$, and those in H_s, H_ℓ, H_{M_ℓ} are assigned high values $> \lceil n/2 \rceil$; see Fig. 5.

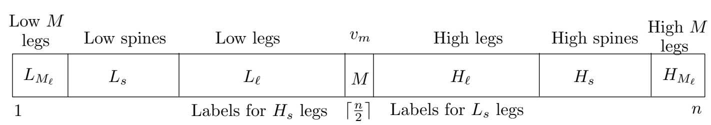


Fig. 5: Allocation of labels to the vertices of a caterpillar graph; legs of L_s are assigned values from the interval H_ℓ and legs of H_s are assigned values from the interval L_ℓ.

Marking: Start by placing all odd-numbered spine vertices in set L_s and all even-numbered ones in H_s, assuming the spine vertices are numbered according to their position in the spine path. The legs of $v \in L_s$ are placed into H_ℓ and the legs of $v \in H_s$ into L_ℓ. Now select *one* vertex to place into set M by traversing the spine vertices $S = L_s \cup H_s$ from right to left. At each vertex $v_i \in S$ we temporarily ignore v_i and its legs from their current sets and check if the following *balance condition* holds

$$|L_s| + |L_\ell| < n/2 \text{ and } |H_s| + |H_\ell| \leq n/2 \tag{3}$$

Intuitively, when the balance condition holds the number of vertices which we will assign to low and high values are both less than $n/2$. If the balance condition holds at $v_i \in S$, we place v_i in set M. Otherwise we *flip* v_i and its legs as follows: if v_i is in set L_s we move it into set H_s and move its legs into L_ℓ else if v_i is in set H_s we move it into set L_s and its legs into H_ℓ.

We claim that the process always stops with a configuration where the balance condition holds. Suppose w.l.o.g that initially $|L_s| + |L_\ell| < n/2$ but $|H_s| + |H_\ell| > n/2$. As vertices and its legs are flipped during the traversal, if the balance condition is not met in the end then we must have $|L_s| + |L_\ell| > n/2$ and $|H_s| + |H_\ell| < n/2$. Hence, at some point, in the traversal, we switch from $|L_s| + |L_\ell| < n/2$ to $|L_s| + |L_\ell| > n/2$ when we flip some vertex v_i and its legs. Ignoring this vertex v_i and its legs ensures that $|L_s| + |L_\ell| < n/2$. Thus we can place vertex v_i into M and stop.

Let v_m be the vertex placed into M. Now we partition the at most Δ legs of v_m into sets L_{M_ℓ} and H_{M_ℓ}, s.t. the total low values, and high values are

$$\text{Low values} \quad |L_s| + |L_\ell| + |L_{M_\ell}| + |M| = \lceil n/2 \rceil \tag{4}$$

$$\text{High values} \quad |H_s| + |H_\ell| + |H_{M_\ell}| = \lfloor n/2 \rfloor \tag{5}$$

Labeling: Label v_m as $\lceil n/2 \rceil$. Then label the legs of v_m in L_{M_ℓ} with values from the interval $[1, |L_{M_\ell}|]$ and its legs in H_{M_ℓ} with values from interval $[n - |H_{M_\ell}| + 1, n]$. As v_m has at most Δ legs, the minimum difference value between v_m and its legs is

$$\min\{\lceil n/2 \rceil - |L_{M_\ell}|, \lceil n/2 \rceil - |H_{M_\ell}|\} \geq n/2 - \Delta \tag{6}$$

For the spine vertices $S = L_s \cup H_s$, assign the vertices of L_s to values in the interval $I(L_s) = [|L_{M_\ell}| + 1, |L_{M_\ell}| + |L_s|]$ and the vertices of H_s to values in the interval $I(H_s) = [n - |H_{M_\ell}| - |H_s| + 1, n - |H_{M_\ell}|]$. Start by assigning values to the spine neighbors of v_m. By the balancing procedure v_m has at most one spine neighbor from L_s and one neighbor from H_s. The neighbor from L_s (if it exists) is assigned $|L_{M_\ell}| + 1$ and the neighbor from H_s (if it exists) is assigned $n - |H_{M_\ell}|$. The difference between v_m and its spine neighbors is

$$\min\{\lceil n/2 \rceil - |L_{M_\ell}|, n - |H_{M_\ell}| - \lceil n/2 \rceil\} \geq n/2 - \Delta - 1 \tag{7}$$

The remaining values of $I(L_s)$ are assigned in increasing order to the remaining spine vertices of L_s. Start with the first vertex in L_s which is left of v_m, then move leftward labeling vertices from L_s until reaching the leftmost vertex in L_s. Now proceed to rightmost vertex in L_s and move leftward again until v_m is reached. Assign the remaining values of the interval $I(H_s)$ to the spine vertices of H_s exactly the same way in an increasing fashion starting from the vertex in H_s to the left of v_m and moving leftward. As we always increment the spine vertices by one, the difference between a spine vertex and its adjacent spine vertices is either

$$\begin{gathered} |L_s| + |L_\ell| + |M| + |H_\ell| \text{ or} \\ |L_s| + |L_\ell| + |M| + |H_\ell| - 1 \end{gathered} \tag{8}$$

In both cases the difference is at least $|L_s| + |L_\ell| + |M| - 1$, which by Equation 4 is at least $n/2 - |L_{M_\ell}| - 1 \geq n/2 - \Delta - 1$.

We now describe how the leg vertices are labeled. The labels of L_ℓ come from interval $I(L_\ell) = [|L_{M_\ell}|+|L_s|+1, \lceil n/2 \rceil - 1]$. The labels of H_ℓ come from the interval $I(H_\ell) = [\lceil n/2 \rceil + 1, \lceil n/2 \rceil + |H_\ell|]$. The values of $I(H_\ell)$ are assigned in increasing order starting with the legs of the spine vertex of L_s with the lowest value. Thus we first label the legs of the spine vertex labeled $|L_{M_\ell}| + 1$, then to legs of $|L_{M_\ell}| + 2$, and so on until $|L_{M_\ell}| + |L_s|$. Assign the values of $I(L_\ell)$ in decreasing order starting with the spine vertex of H_s with the highest label. Thus the legs of $n - |H_{M_\ell}|$ are labeled first, then the legs of $n - |H_{M_\ell}| - 1$ and so on until $n - |H_{M_\ell}| - |H_s| + 1$.

As all spines have at least one leg, the difference between the j-th lowest spine vertex of L_s and one of its legs is at least the value given by Equation 9 (same for H_s):

$$\begin{aligned} |L_s| - j + |L_\ell| + |M| + j &\geq \lceil n/2 \rceil - |L_{M_\ell}| - 1 \text{ by Equation 4} \\ |H_s| - j + |H_\ell| + |M| + j &\geq \lceil n/2 \rceil - |H_{M_\ell}| - 1 \text{ by Equation 5} \end{aligned} \tag{9}$$

As $|L_{M_\ell}|$ and $|H_{M_\ell}|$ are both $\leq \Delta$ the differences are at least $n/2 - \Delta - 1$.

5.2 Extending to general caterpillars

We extend the labeling scheme to general caterpillar G to achieve differential value at least $n/2 - \Delta - 2$. The main idea is to consider spine vertices with no legs as *pseudo-legs* of their neighbors, thus transforming G into a caterpillar where all but one spine vertex has at least one leg or pseudo-leg. See Figure 6. Observe that the right-most spine vertex of G has at least one leg as otherwise it is a leg of the spine vertex to its left.

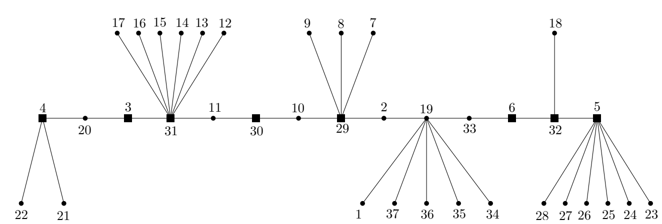


Fig. 6: Labeling of a general caterpillar, where the spine vertices drawn as circles correspond to pseudo legs.

Marking: Select a vertex v_m for set M as before. Then traverse the spine from left to right to determine which vertices are pseudo-legs. Let v be the current spine vertex and v' be the spine vertex to the right of v. If $v \neq v_m$ and v currently has no legs or pseudo-legs then first assign v to be a pseudo-leg of v' and move v into the corresponding set as follows: if $v' \in L_s$ move v into set H_ℓ, if $v' \in H_s$ then move v into set L_ℓ, and if $v' \in M$ then keep v in its current set. Observe that in the first case vertex v moves from set H_s into set H_ℓ, and in the second it moves from set L_s into set L_ℓ. Thus the number of low and high values to be assigned both remain $< n/2$ and balance condition in Equation 3 is still satisfied.

Now let v'_m be the right neighbor of v_m on the spine. If v'_m is currently a pseudo-leg reassign v'_m to be a pseudo-leg of v_m, else if $v'_m \in L_\ell$ move v'_m into L_s and if $v'_m \in H_\ell$

move it into H_s. Note that the balance condition in Equation 3 is maintained. However as we reassigned v'_m to be a pseudo-leg of v_m this may leave one vertex, namely the spine vertex to the right of v'_m with no legs. Finally partition the *real* legs of v_m into the sets L_{M_ℓ}, H_{M_ℓ} as before.

Labeling: Label v_m as $\lceil n/2 \rceil$. Then label v_m's *real* legs and its neighboring vertices on the spine also as before. Note that now the two neighboring spine vertices of v_m may be pseudo-legs of v_m. Still, one of them is in the set L_s and the other is in the set H_s, and we label these as $|L_{M_\ell}| + 1$ and $n - |H_{M_\ell}|$ respectively. Equations 6 and 7 still apply so the differential value for v_m is at least $n/2 - \Delta - 1$.

Finally label the remaining spine vertices from $S = L_s \cup H_s$ and their legs and pseudo-legs $L_\ell \cup H_\ell$ as before. We show that the difference between adjacent vertices is at least $n/2 - \Delta - 2$. First consider two adjacent vertices from S. Their difference is still given by Equation 8 and is at least $|L_s| + |L_\ell| + |M| - 1$. By Equation 4 this is at least $n/2 - |L_{M_\ell}| - 1$, and as $|L_{M_\ell}| \leq \Delta$, the difference is $\geq n/2 - \Delta - 1$.

Next consider a vertex of $v \in S$ and its legs (including its pseudo-leg). We modify Equation 9 to take into account that at most one spine vertex may have no legs. The difference between the j-th lowest spine vertex of L_s and one of its legs and the j-th lowest spine of H_s and ones of its legs is at least:

$$|L_s| - j + |L_\ell| + |M| + (j-1) \geq \lceil n/2 \rceil - |L_{M_\ell}| - 1 \text{ by Equation 4} \quad (10)$$
$$|H_s| - j + |H_\ell| + |M| + (j-1) \geq \lceil n/2 \rceil - |H_{M_\ell}| - 1 \text{ by Equation 5} \quad (11)$$

As $|L_{M_\ell}|$ and $|H_{M_\ell}|$ are both $\leq \Delta$ the difference is at least $n/2 - \Delta - 1$.

Finally consider a vertex $v \in S$ which is adjacent to v_p where v_p is a pseudo-leg of $v' \in S$, $v' \neq v$. Each vertex $v \in S$ may be adjacent to at most one such v_p. As v' may have v_p as its only leg and as there is at most one spine vertex with no legs the difference depending on the label of v is at least

$$|L_s| - j + |L_\ell| + |M| + (j-2) \geq \lceil n/2 \rceil - |L_{M_\ell}| - 2 \text{ by Equation 4} \quad (12)$$
$$|H_s| - j + |H_\ell| + |M| + (j-2) \geq \lceil n/2 \rceil - |H_{M_\ell}| - 2 \text{ by Equation 5} \quad (13)$$

Again as $|L_{M_\ell}|$ and $|H_{M_\ell}|$ are $\leq \Delta$, the difference is at least $n/2 - \Delta - 2$, proving Theorem 6. □

5.3 Comparison with the Miller-Pritikin scheme

Consider a non-regular caterpillar with $s = 2k + 1$ spine vertices, where the $k + 1$ odd spine vertices have 1 leg each and the k even spine vertices have Δ legs each. This forms a bipartite graph with disjoint vertex sets U and V, where the $k + 1$ odd spine vertices and the $k\Delta$ legs of even spine vertices form the set U, and the rest of the vertices form set V. The Miller-Pritikin labeling achieves differential value equal to the size of the smaller vertex set i.e., $\min\{|U|, |V|\} = 2k+1$. On the same graph our labeling scheme gives value at least $n/2 - \Delta - 1 = \frac{2k+1+(k+1)+(\Delta k)}{2} - \Delta - 1 \geq \frac{k}{2}(\Delta - 1)$. Let $\Delta = \Omega(n)$ and $k = O(1)$. Then the Miller-Pritikin scheme achieves differential value $O(1)$ whereas our labeling achieves $O(n)$, making it potentially worse than ours by a factor of $\Omega(n)$.

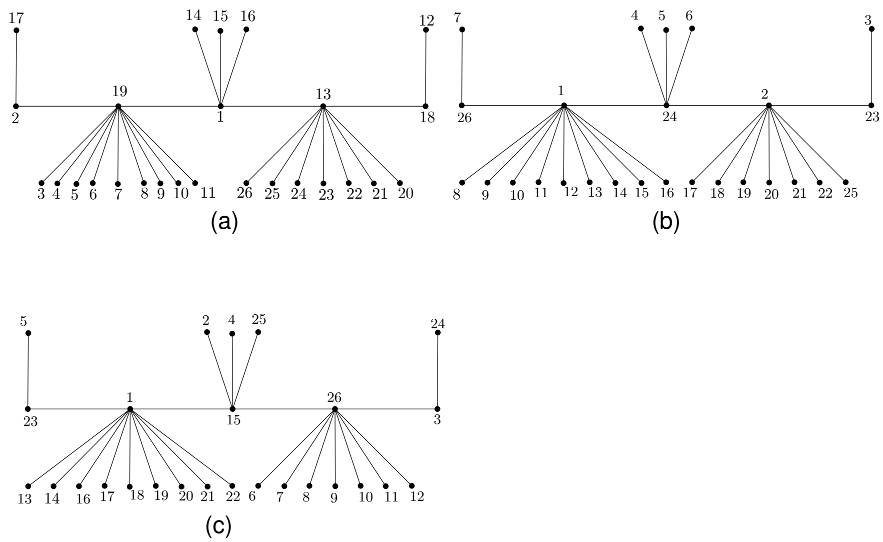


Fig. 7: Three different labelings of a caterpillar graph. (a) The labeling given by Theorem 6 achieves differential $18 - 13 = 5$. (b) The Miller-Pritikin labeling achieves differential 7. (c) A manually generated labeling achieving differential $25 - 15 = 10$.

There are also graphs for which Miller-Pritikin scheme achieves differential value better than the value our labeling achieves. Figure 7 gives an example of a caterpillar labeled in three different ways; by our labeling of Theorem 6, by the Miller-Pritikin labeling and manually generated. The labeling of Theorem 6 achieves the lowest differential value while the Miller-Pritikin labeling is only slightly better and the manually generated labeling is twice as good as the labeling of Theorem 6.

6 Conclusion and Future Work

We proved new upper bounds for the maximum differential coloring problem of caterpillars and spiders and proved optimality of the Miller-Pritikin scheme for regular caterpillars and spiders. Still, for general caterpillars neither ours nor the Miller-Pritikin labeling scheme is optimal. We show that our labeling scheme is guaranteed to be off from the optimal labeling (as well as the Miller *et al.* labeling) by at most an additive value of $\Delta + 2$. Nevertheless there are instances where the Miller-Pritikin scheme performs considerable worse than our scheme (Section 5.3). There exist several natural open problems including finding optimal labeling schemes, proof of NP hardness or good approximations for various classes of graphs, such as general caterpillars, lobsters, trees, interval graphs, cubic graphs, regular bipartite graphs.

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