CSc 120
Introduction to Computer Programming II

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08: Efficiency and Complexity


EFFICIENCY MATTERS

reasoning about performance

Reasoning about efficiency

- Not *just* the time taken for a program to run
  - this can depend on:
    - processor properties that have nothing to do with the program *(e.g., CPU speed, amount of memory)*
    - what other programs are running *(i.e., system load)*
    - which inputs we use *(some inputs may be worse than others)*

- We would like to compare different algorithms:
  - without requiring that we implement them both first
  - abstracting away processor-specific details
  - considering all possible inputs

Reasoning about efficiency

- Algorithms vs. programs
  - Algorithm:
    - a step-by-step list of instructions for solving a problem
  - Program:
    - an algorithm that been implemented in a given language
- We would like to compare different algorithms *abstractly*

Comparing algorithms

- Search for a word `my_word` in a dictionary (a book)
- *A dictionary is sorted*
  - Algo 1:
    start at the first word in the dictionary
    if the word is not `my_word`, then go to the next word
    continue in sequence until `my_word` is found

  - Algo 2:
    start at the middle of the dictionary

    if `my_word` is greater than the word in the middle,

        start with the middle word and continue from

        there to the end

    if `my_word` is less than the word in the middle,
        start with the middle word and continue from
        there to the beginning

Comparing algorithms

- Which is better, Algo 1 or Algo 2?

  Algo 2 in most cases (seemingly)

  What is the reason?

- When is Algo 1 better?

  Algo 1 is better if the word is close to the beginning

  How close to the beginning?

- When considering which is better, what measure are we using?

  The number of comparisons

Comparing algorithms

- Call comparison a *primitive* operation
  - an abstract unit of computation

- We want to characterize an algorithm in terms of how many primitive operations are performed
  - best case and worst case

- We want to express this in terms of the size of the data (or size of its input)

Primitive operations

- Abstract units of computation
  - convenient for reasoning about algorithms
  - approximates typical hardware-level operations
- Includes:
  - assigning a value to a variable
  - looking up the value of a variable
  - doing a single arithmetic operation
  - comparing two numbers
  - accessing a single element of a Python list by index
  - calling a function
  - returning from a function

Primitive ops and running time

- A primitive operation typically corresponds to a small constant number of machine instructions
- No. of primitive operations executed

  $\propto$ no. of machine instructions executed

  $\propto$ actual running time

- We consider how a function's running time depends on the size of its input
  - *which input do we consider?*

Best case vs. worst case inputs

```
# lookup(str_, list_): returns the index where str_ occurs in list_
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

- Best-case scenario: str_ == list_[0]   *# first element*
  - loop does not have to iterate over list_ at all
  - running time does not depend on length of list_
  - does not reflect typical behavior of the algorithm

Best case vs. worst case inputs

```
# lookup(str_, list_): returns the index where str_ occurs in list_

def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

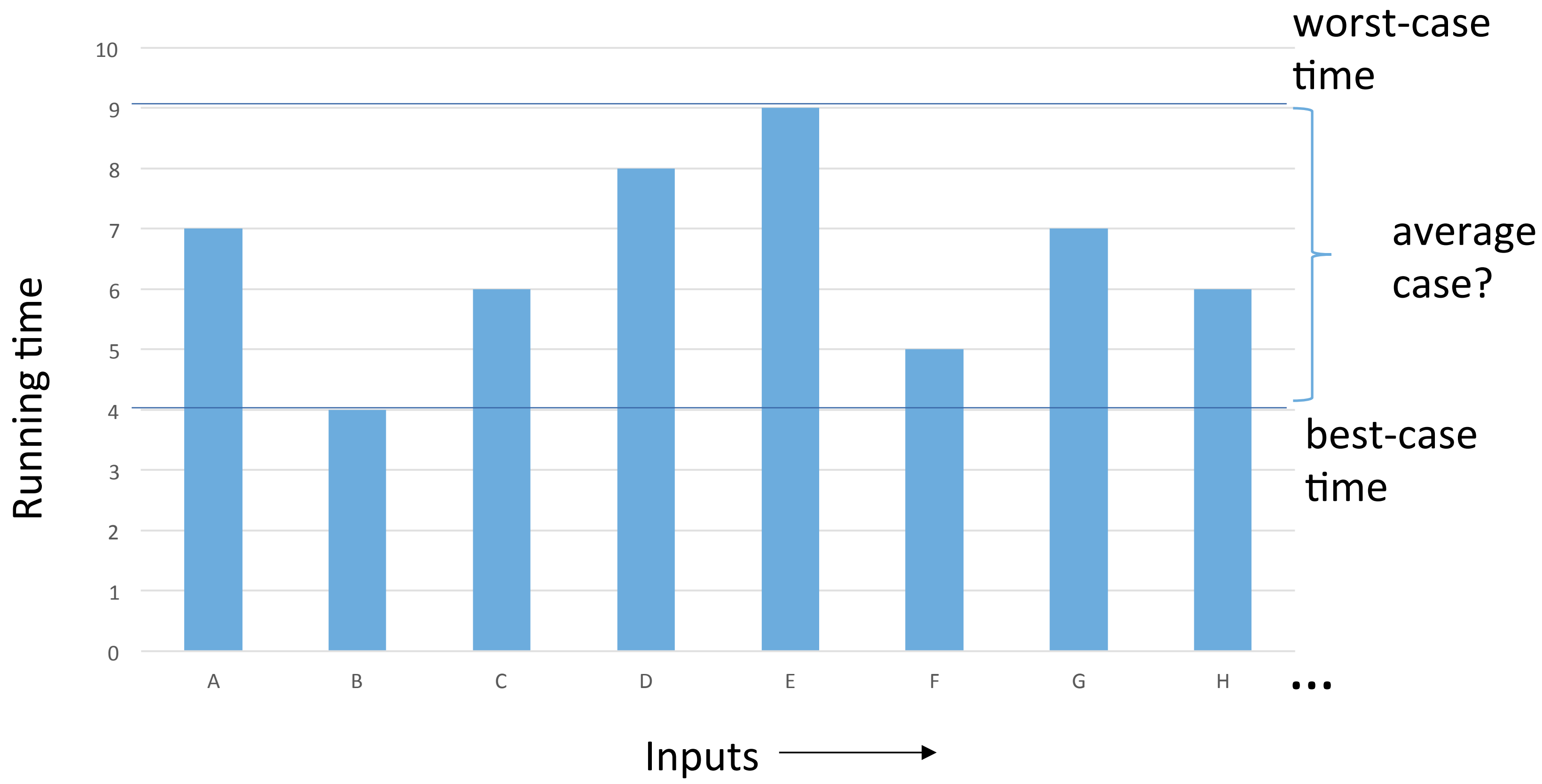
- Worst-case scenario: str_ == list_[-1]   *# last element*
  - loop iterates through list_
  - running time is proportional to the length of list_
  - captures the behavior of the algorithm better

Best case vs. worst case inputs

```
# lookup(str_, list_): returns the index where str_ occurs in list_
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

- In reality, we get something in between
  - but "average-case" is difficult to characterize precisely

What about “average case”?



Worst-case complexity

- Considers worst-case inputs
- Describes the running time of an algorithm as a function of the size of its input ("time complexity")
- Focuses on the *rate* at which the running time grows as the input gets large
- Typically gives a better characterization of an algorithm's performance

- This approach can also be applied to the amount of memory used by an algorithm ("space complexity")

Example

Code

```
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

Primitive operations

len(list_) :	1
range( ) :	1
in :	1
for :	2
list_[i] :	1
str_ :	1
== :	1
if :	1

each iteration:
9 primitive ops

Example

Code

```
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

Primitive operations

len(list_) :	1
range( ) :	1
in :	1
for :	2
list_[i] :	1
str_ :	1
== :	1
if :	1

each iteration:
9 primitive ops

Total primitive ops executed:

1 iteration: 9 ops

∴ n iterations: 9n ops

+ return at the end: 1 op

∴ total worst-case running time for a list of length n = 9n + 1

EXERCISE

 # What is the total worst-case running time of the following code fragment expressed in terms of n?

```
a = 5
b = 10
for i in range(n):
    x = i * b
for j in range(n):
    z += b
```

asymptotic complexity

Asymptotic complexity

- In the worst-case, lookup(str_, list_) executes 9n + 1 primitive operations given a list of length n
- To translate this to running time:
  - suppose each primitive operation takes *k* time units
  - then worst-case running time is (9n + 1)*k*
- But *k* depends on specifics of the computer, e.g.:

Processor speed	***k***	**running time**
slow	20	180n + 20
medium	10	90n + 10
fast	3	27n + 3

Asymptotic complexity

depends on processor-specific characteristics

worst case running time = $\overbrace{A}n + \overbrace{B}$

$A\underbrace{n} + B$

depends on how the algorithm processes data

Asymptotic complexity

- For algorithm analysis, we focus on how the running time grows as a function of the input size *n*
  - usually, we do not look at the exact worst case running time
  - it's enough to know proportionalities

- E.g., for the lookup() function:
  - we say only that its running time is *"proportional to the input length n"*

Example

Code

```
def list_positions(list1, list2):
    positions = []
    for value in list1:
        idx = lookup(value, list2)
        positions.append(idx)
    return positions
```

Example

Code | **Primitive operations**

```
def list_positions(list1, list2):
    positions = []                     1
    for value in list1:                in :   1
                                       for :  2
        idx = lookup(value, list2)     9n + 1
        positions.append(idx)          1
    return positions                   1
```

(in, for, lookup and append lines: iterates n times)

Worst case behavior:

primitive operations $= n(9n + 5) + 2 = 9n^2 + 5n + 2$

running time $= k(9n^2 + 5n + 2)$

Example

Code

```
def list_positions(list1, list2):
    positions = []
    for value in list1:
        idx = lookup(value, list2)
        positions.append(idx)
    return positions
```

Worst case: $\underbrace{9n^2} + \underbrace{5n + 2}$

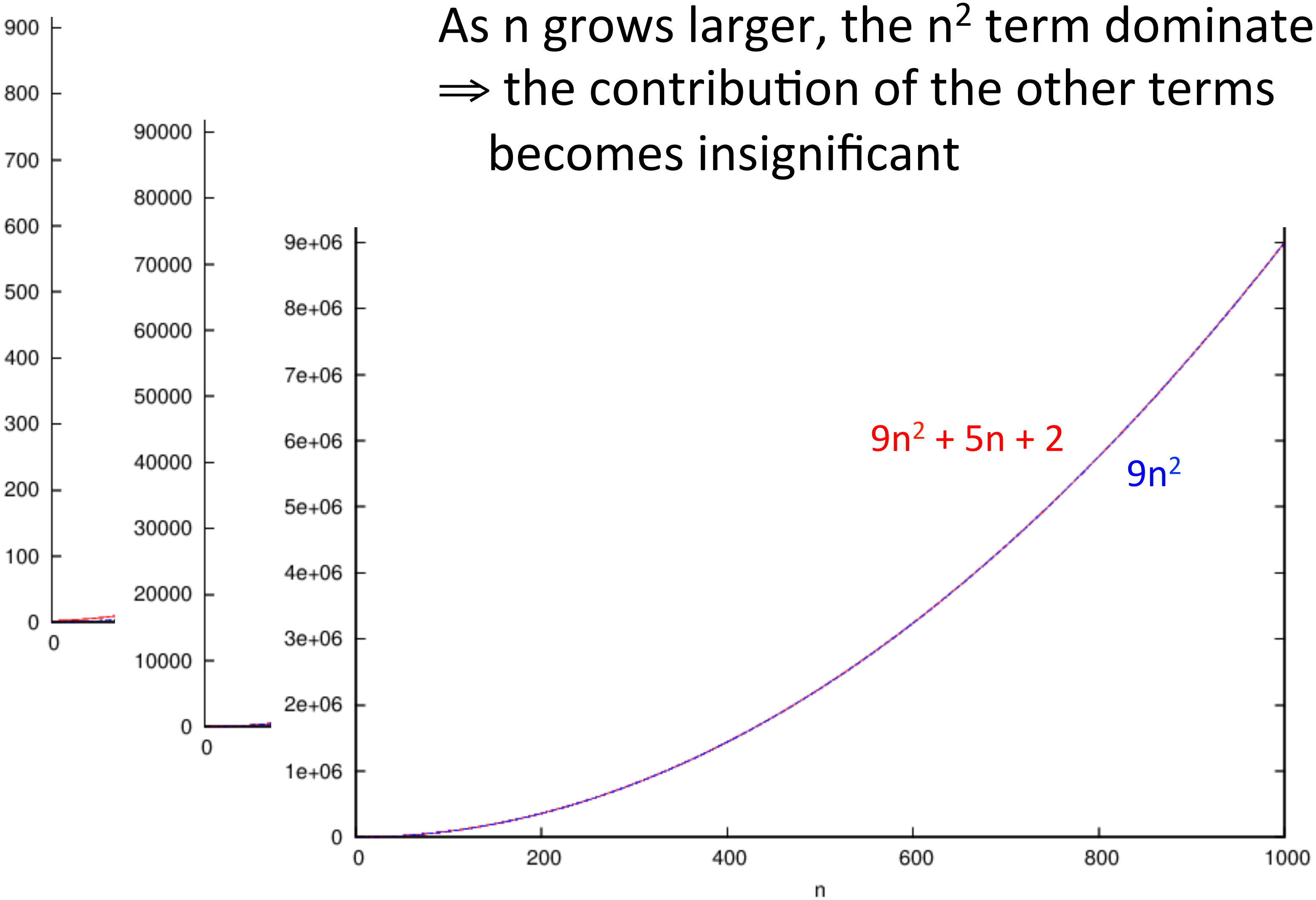
As n grows, the $9n^2$ term grows faster than $5n+2$

$\Rightarrow$ for large n, the n^2 term dominates

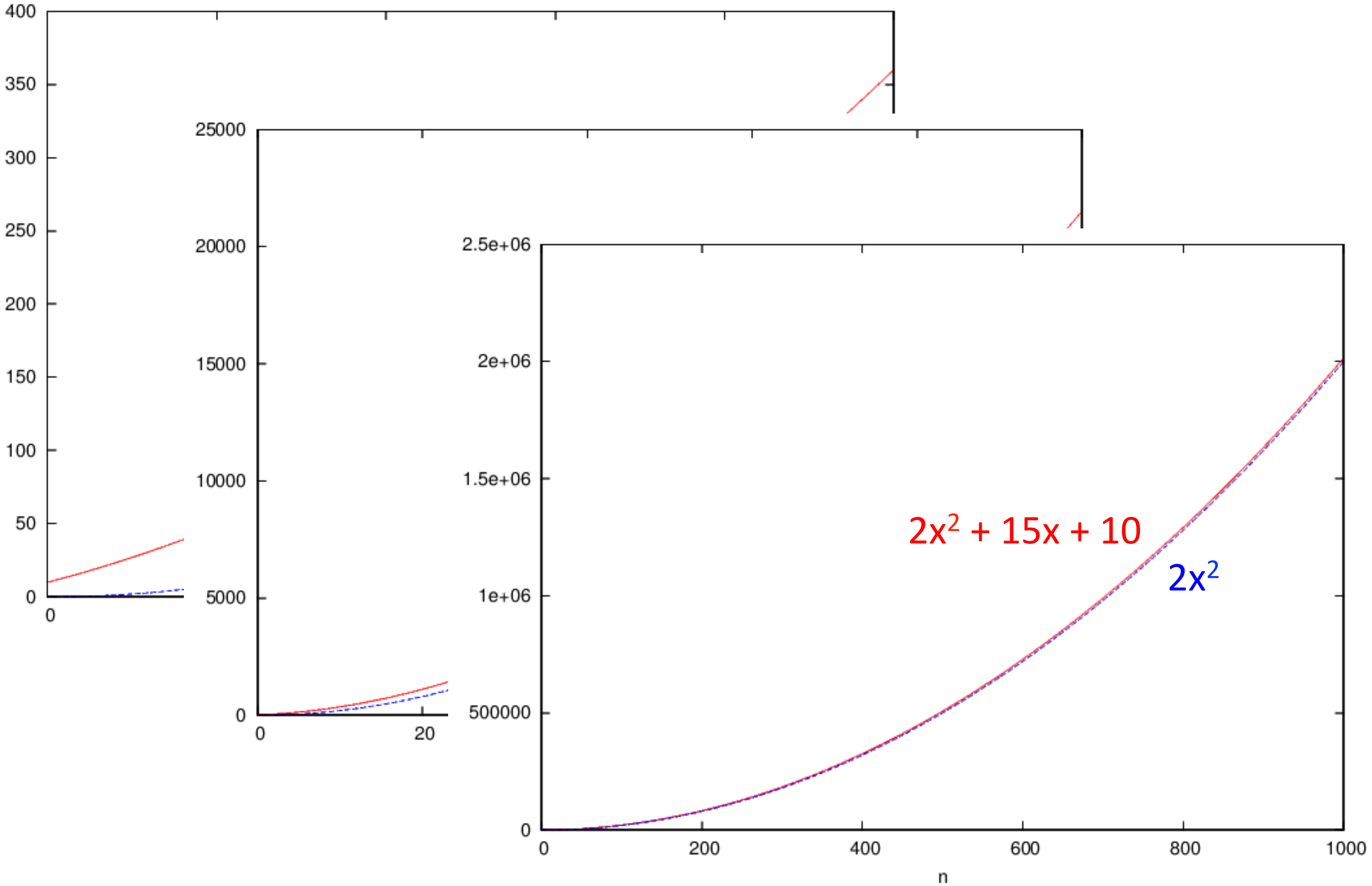
$\Rightarrow$ running time depends primarily on n^2

Example

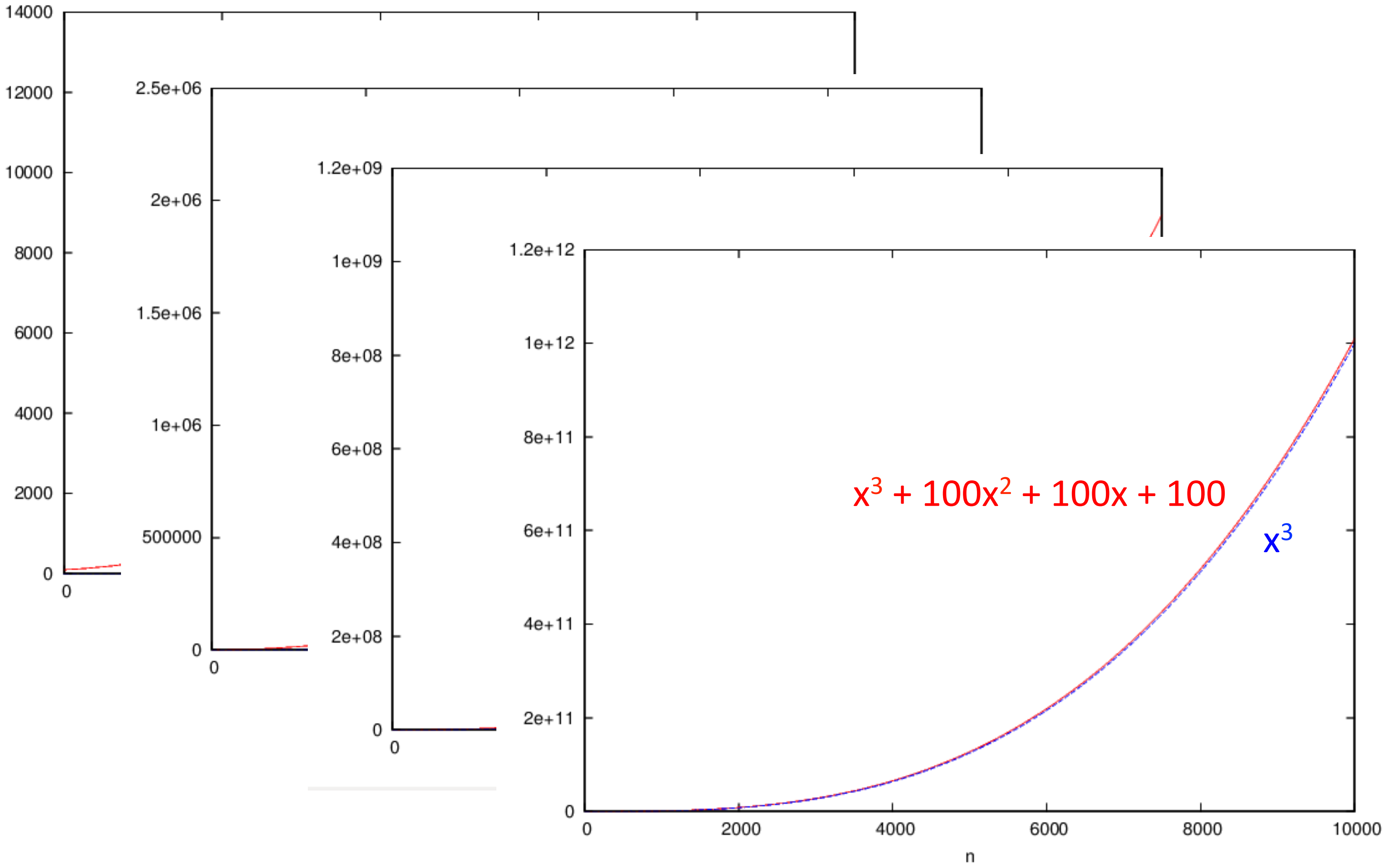
As n grows larger, the n^2 term dominates
$\Rightarrow$ the contribution of the other terms becomes insignificant



Example 2: $2x^2 + 15x + 10$



Example 3: $x^3 + 100x^2 + 100x + 100$



Growth rates

- As input size grows, the fastest-growing term dominates the others
  - the contribution of the smaller terms becomes negligible
  - it suffices to consider only the highest degree (i.e., fastest growing) term

- For algorithm analysis purposes, the constant factors are not useful
  - they usually reflect implementation-specific features
  - to compare different algorithms, we focus only on proportionality

  $\Rightarrow$ ignore constant coefficients

Comparing algorithms

Growth rate ∝ n

```
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

Growth rate ∝ n²

```
def list_positions(list1, list2):
    positions = []
    for value in list1:
        idx = lookup(value, list2)
        positions.append(idx)
    return positions
```

Summary so far

- Want to characterize algorithm efficiency such that:
  - does not depend on processor specifics
  - accounts for all possible inputs

  ⇒ count primitive operations

  ⇒ consider worst-case running time

- We specify the running time as a function of the size of the input
  - consider proportionality, ignore constant coefficients
  - consider only the dominant term
    - e.g., $9n^2 + 5n + 2 \approx n^2$

big-O notation

Big-O notation

- Big-O is formalizes this intuitive idea:
  - consider only the dominant term
    - e.g., $9n^2 + 5n + 2 \approx n^2$
  - allows us to say,
    "the algorithm runs in time proportional to n^2"

Big-O notation

Intuition:

When we say…	*…we mean*
"f(n) is O(g(n))"	"f is growing at most as fast as g"

"big-O notation"

Big-O notation

- Captures the idea of the growth rate of functions, focusing on proportionality and ignoring constants

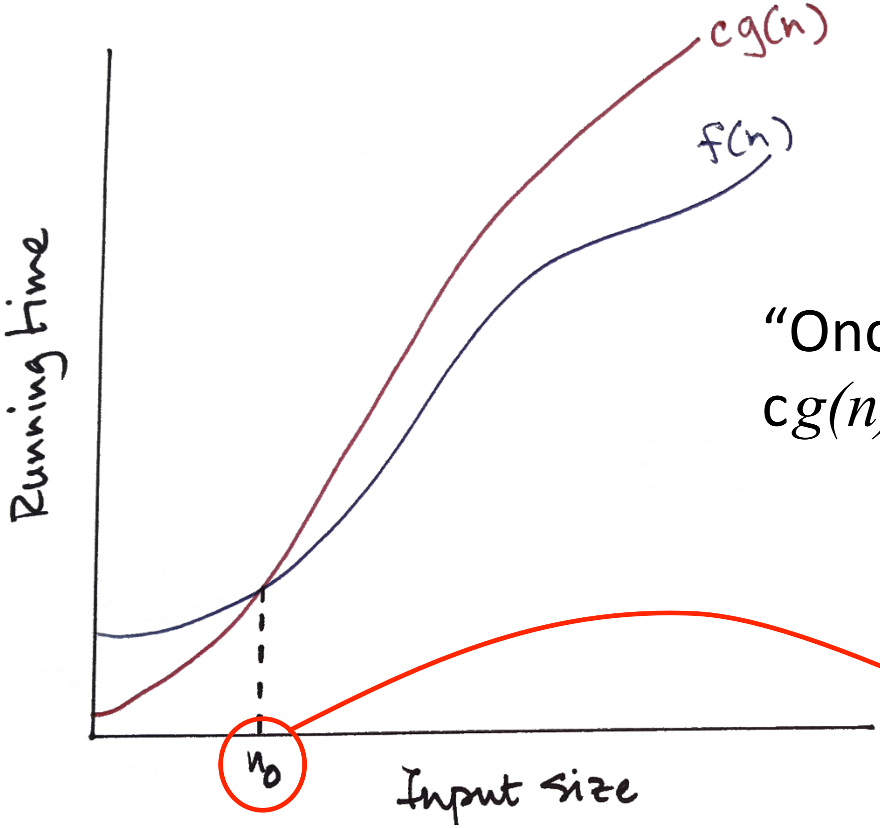
Definition: Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers.

Then, $f(n)$ is $O(g(n))$ if there is a real constant c and an integer constant $n_0 \geq 1$ such that

$$f(n) \leq c\,g(n) \quad \text{for all } n > n_0$$

Big-O notation

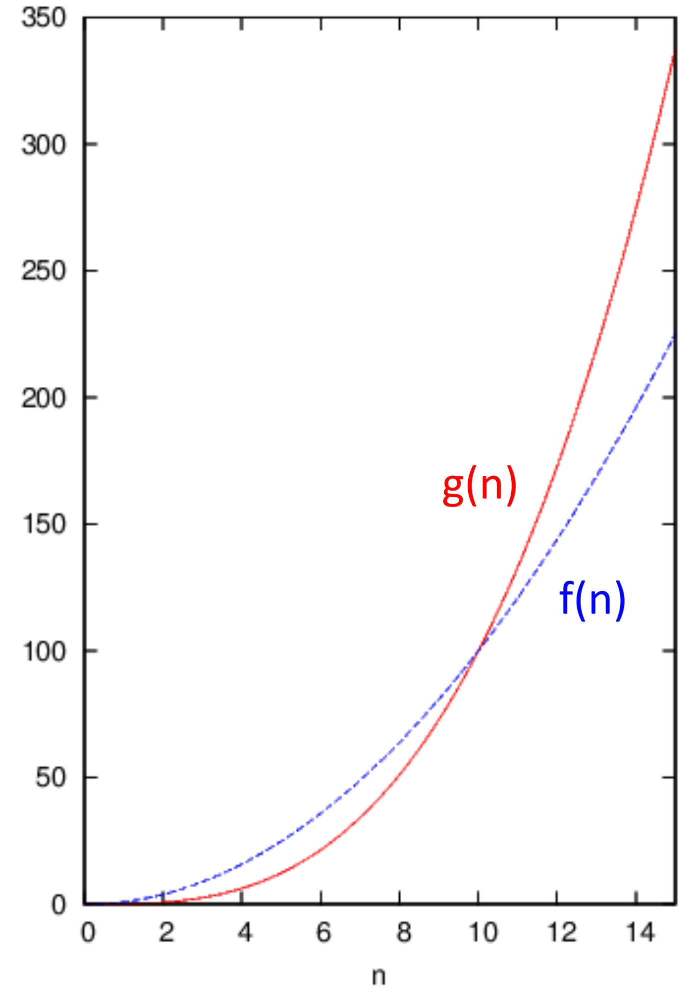
$f(n)$ is O($g(n)$) if there is a real constant c and an integer constant $n_0 \geq 1$ such that $f(n) \leq c\, g(n)$ for all $n > n_0$



"Once the input gets big enough, c$g(n)$ is (always) larger than $f(n)$"

Big-O notation: properties

- If g(n) is growing faster than f(n):
  - f(n) is O(g(n))
  - g(n) is not O(f(n))

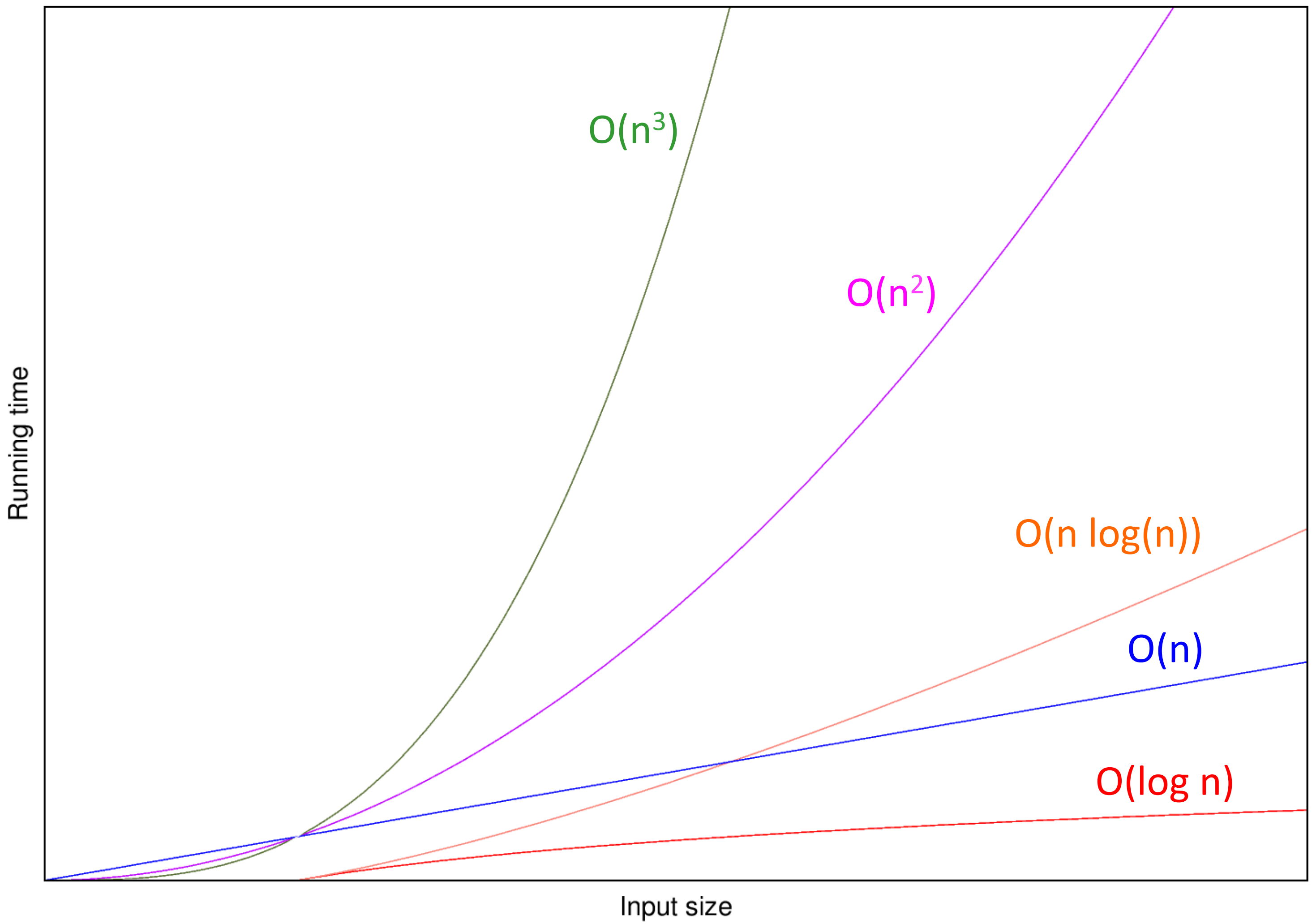


- If f(n) = $a_0 + a_1 n + \ldots + a_k n^k$, then:

$$f(n) = O(n^k)$$

  - i.e., coefficients and lower-order terms can be ignored

Some common growth-rate curves



using big-O notation

Computing big-O complexities

Given the code:

line$_1$ ... $O(f_1(n))$
line$_2$ ... $O(f_2(n))$
...
line$_k$ ... $O(f_k(n))$

The overall complexity is

$O(\max(f_1(n), f_s(n), \ldots, f_k(n)))$

Given the code

loop ... O(f1(n)) iterations
    line1 ... O(f2(n))

The overall complexity is

$O(\, f_1(n) \times f_2(n) \,)$

Using big-O notation

Code	Big-O complexity
str_ == list_[i]	O(1)

O(1)

O(1)

O(1)

Using big-O notation

Code	Big-O complexity

O(1)
`if str_ == list_[i]:`
`return i`
O(1)
O(1)
	O(1)

Using big-O notation

Code	Big-O complexity

`for i in range(len(list_)):`
`    if str_ == list_[i]:`
`        return i`
	O(n)

O(n) (worst-case)
(n = length of the list)

O(1)

Using big-O notation

Code	Big-O complexity
see below	O(n)

```
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

O(n)

O(1)

Using big-O notation

Code	Big-O complexity
for value in list1: idx = lookup(value, list2)	$O(n^2)$

O(n) (worst-case)
(n = length of list1)

O(n) (worst-case)
(n = length of list2)

Using big-O notation

Code	Big-O complexity

def list_positions(list1, list2):
    positions = []
    for value in list1:
        idx = lookup(value, list2)
        positions.append(idx)
    return positions
	$O(n^2)$

O(1)

$O(n^2)$

Computing big-O complexities

Given the code:

line$_1$ ... $O(f_1(n))$
line$_2$ ... $O(f_2(n))$
...
line$_k$ ... $O(f_k(n))$

The overall complexity is

$O(\max(f_1(n), f_s(n), ..., f_k(n)))$

Given the code

loop ... O(f1(n)) iterations
    line1 ... O(f2(n))

The overall complexity is

$O(\ f_1(n) \text{ x } f_2(n)\ )$

EXERCISE

```
# my_rfind(mylist, elt) : find the distance from the
# end of mylist where elt occurs, -1 if it does not
def my_rfind(mylist, elt):
    pos = len(mylist) – 1
    while pos >= 0:
        if mylist[pos] == elt:
            return pos
        pos -= 1
    return -1
```

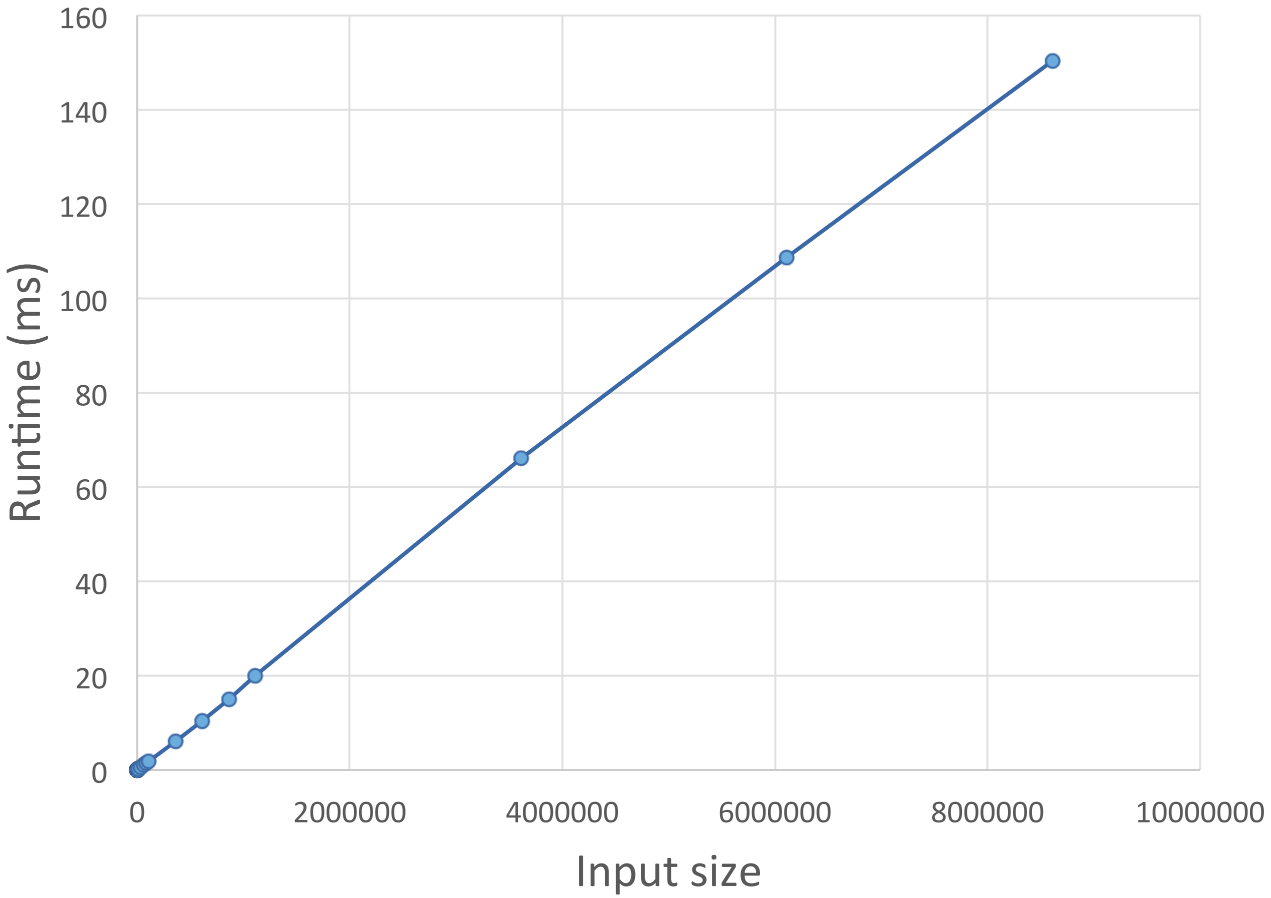
Worst-case big-O complexity = ???

EXERCISE

```
# for each element of a list: find the biggest value
# between that element and the end of the list
def find_biggest_after(arglist):
    pos_list = []
    for idx0 in range(len(arglist)):
        biggest = arglist[idx0]
        for idx1 in range(idx0+1, len(arglist)):
            biggest = max(arglist[idx1], biggest)
        pos_list.append(biggest)
    return pos_list
```

Worst-case big-O complexity = ???

Input size vs. run time: max()



EXERCISE

```
# for each element of a list: find the biggest value
# between that element and the end of the list
def find_biggest_after(arglist):
    pos_list = []
    for idx0 in range(len(arglist)):
        biggest = max(arglist[idx0:])   # library code
        pos_list.append(biggest)
    return pos_list
```

Worst-case big-O complexity = ???

WARM-UP

What is the worst case running time of the following function?

$$4n^2 + 5n + 2$$

Why can we ignore the constants and lower order terms?

Is analyzing worst-case running time important?

How many Web pages are there?