CSc 120
Introduction to Computer Programming II

15: Hashing

Hashing

Searching

We have seen two search algorithms:

- linear (sequential) search O(n)
  - the items are not sorted
- binary search O(log n)
  - the items are sorted
  - must consider the cost of sorting

- Can we do better?
- Have you considered how a Python dictionary might be implemented?

ADT - Dictionary

- A dictionary is an ADT that holds key/value pairs and provides the following operations:
  - put(key, value)
    - makes an entry for a key/value pair
    - assumes key is not already in the dictionary
  - get(key) looks up key in the dictionary
    - returns the value associated with key (and None if not found)

ADT - Dictionary

Usage:

```
>>> d = Dictionary(7)
>>>
>>> d.put('five', 5)
>>> d.put('three', 3)
```

Problem:

Implement Dictionary

Hint:

```
>>> d._pairs
[['five', 5], ['three', 3], None, None, None, None, None]
```

ADT – Dictionary solution 1

```
class Dictionary:
    def __init__(self,capacity):
        # each element will be a key/value pair
        self._pairs = [None] * capacity
        self._nextempty = 0

    def put(self, k, v):
        self._pairs[self._nextempty] = [k,v]
        self._nextempty += 1

    def get(self, k):
        for pair in self._pairs[0:self._nextempty]:
            if pair[0] == k:
                return pair[1]
        return None
```

Performance

- What is big-O of the Dictionary's methods?
  - put()
  - get()

- Can we do better than O(n) for get()?
- Consider this:

  alist[3] # this "get" or "lookup" is O(1)

- Why is this O(1)?

  indices are contiguous

  easy to compute starting point plus offset

- Can we 'transform' keys into integers that fall into a small, contiguous range?

Beating O(n)

Can we 'transform' keys into integers that fall into a small range?

"hello" -> 147

"a" -> 422

How could we turn a key (string) into an integer?

- simple method: use the length

“Hash” the key (colloquial meaning)

Chop up the key

Scramble the key to get a value

Hashing

- A hash function is a function that can be used to map data of arbitrary size to a value in a fixed range

- Is the following a hash function?

```
def hash(key):
    return len(key)
```

- Strings are arbitrary length
  - modify `hash(key)` to return a value in a fixed range
  - an integer between 0 and 7

Exercise

Problem:

Modify Dictionary to use a hash function to compute the index for a new key/value pair.

(See solution on slide 28.)

Hashing

What happens in this situation?

```
>>> d.put('hello', 14)
>>> d.put('e', 351)
>>> d.put('hat', 8)
>>> d.put('conciousness', 1)
```

Hashing

- Hash results:

key	hash value
'hello'	5
'e'	1
'hat'	3
'consciousness'	5

- *Collision*: two or more keys have the same hash value

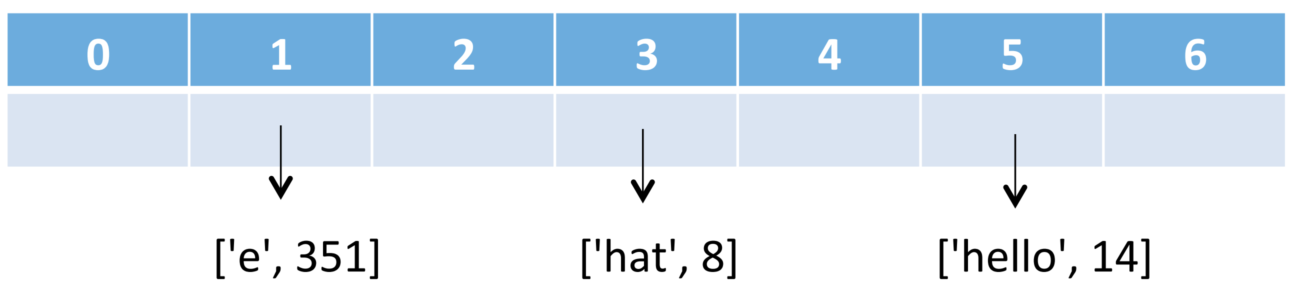
Hashing

- Hash results:

key	hash value
'hello'	5
'e'	1
'hat'	3
'consciousness'	5

collision

- Dictionary implementation view:

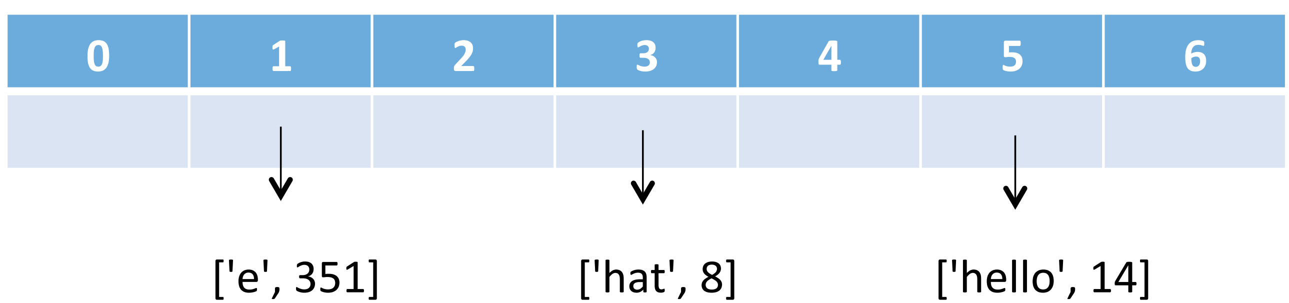


Need a place to put ['consciousness', 1]

Hashing and collisions

- *perfect hash function*: every key hashes to a unique value
  – most hash functions are not perfect

- Need a systematic method for placing keys in a Dictionary (hash table) when collisions occur.



Need a place to put ['consciousness', 1]

Collision Resolution

- Methods for resolving collisions:
  - increase the table size (the list in our example)

    consider social security numbers: 333-55-8888

    9 digits / 10^9 entries

  - *open addressing*
    - compute the hash value
    - on collision, sequentially visit each slot in the hash table to find an available spot
    - visit each slot by going 'lower' in the table (decrement by 1)
    - wrap if necessary

Collision Resolution

- Simplify the example by using integers for keys
- Hash function

  h(key) = key % 7

- Hash values for the keys: 14, 2, 10, 19

key	hash value
14	0
2	2
10	3
19	5

- Hash table

0	1	2	3	4	5	6
14		2	10		19	

Collision Resolution

- keys: 14, 2, 10, 19
- Now add 24
  - h(key) = key % 7
    = 24 % 7
    = 3 ← collision, use open addressing
- Hash table

0	1	2	3	4	5	6
14		2	10		19	

↑

h(24) = 3 – collision

Collision Resolution

- keys: 14, 2, 10, 19
- Now add 24
  - h(key) = key % 7
    = 24 % 7
    = 3 ← collision, use open addressing
- Hash table

0	1	2	3	4	5	6
14		2	10		19	

h(24) = 3 – collision

look lower – occupied

Collision Resolution

- keys: 14, 2, 10, 19
- Now add 24
  - h(key) = key % 7

    = 24 % 7

    = 3 ← collision, use open addressing
- Hash table

0	1	2	3	4	5	6
14		2	10		19	

h(24) = 3 – collision

look lower – occupied

look lower – empty

Collision Resolution

- *Probe sequence*: the locations examined when inserting a new key

  h(24) = 3

- The hash computation is the first "probe"
- Hash table

0	1	2	3	4	5	6
14		2	10		19	

Collision Resolution

- *Probe sequence*: the locations examined when inserting a new key

  h(24) = 3

- The hash computation is the first "probe"
- Hash table

0	1	2	3	4	5	6
14		2	10		19	

first probe – collision 3

Collision Resolution

- *Probe sequence*: the locations examined when inserting a new key

  h(24) = 3

- The hash computation is the first "probe"
- Hash table

0	1	2	3	4	5	6
14		2	10		19	

first probe – collision 3

second probe – occupied 2

Collision Resolution

- *Probe sequence*: the locations examined when inserting a new key

  h(24) = 3

- The hash computation is the first "probe"
- Hash table

0	1	2	3	4	5	6
14		2	10		19	

first probe – collision 3

second probe – occupied 2

third probe – empty 1

Collision Resolution

- *Probe sequence*: the locations examined when inserting a new key

  $h(24) = 3$

- The hash computation is the first "probe"
- Hash table

probe sequence: 3, 2, 1

0	1	2	3	4	5	6
14	**24**	2	10		19	

first probe – collision 3

second probe – occupied 2

third probe – empty 1

Exercise

Use open addressing to insert the key 23 into the hash table below. Give the probe sequence.

The hash function is the key % 7

hash table

0	1	2	3	4	5	6
14	24	2	10		19	

Collision Resolution

open addressing:

- the probe sequence is linear
- the probe decrement is 1

open addressing with *linear probing* has serious performance problems (!!)

When two keys collide at the same hash value, they will follow the same initial probe sequence

Can we do better?

Hint: change the probe decrement.

Hashing

- SHA-1 (Secure Hash Algorithm 1)
  - cryptographic hash function designed by the NSA
  - 120 bits
  - shown as hexadecimal number, 40 digits long

  https://wingware.com/downloads/wingide-101

- MD5 (Message Digest 5)
  - widely used hash function to verify data integrity
  - now compromised
  - 128 bits

  http://archive.eclipse.org/eclipse/downloads/drops/R-3.8.2-201301310800/

ADT – Dictionary solution w/hashing

```
class Dictionary:
  def __init__(self, capacity):
    # each element will be a key/value pair
     self._pairs = [None] * capacity

  def _hash(self, k):
     return len(k) % len(self._pairs)

  def put(self, k, v):
     self._pairs[self._hash(k)] = [k,v]      #use the hash function

  def get(self, k):
     return self._pairs[self._hash(k)][1] #use the hash function
```

Questions

What is a hash function?

What is a collision?

In open addressing with linear probing, how are collisions resolved?

Collision Resolution (revisited)

open addressing

- *open addressing with linear probing*
  - compute the hash value
  - on collision, sequentially visit each slot in the hash table to find an available spot
  - visit each slot by going 'lower' in the table (decrement by 1)
  - wrap if necessary

terminology

- the probe sequence is linear
- the probe decrement is 1

Collision Resolution (revisited)

- keys: 14, 2, 10, 19
- Now add 24
  - h(key) = key % 7
    = 24 % 7
    = 3 ← collision, use open addressing
- Hash table

0	1	2	3	4	5	6
14		2	10		19	

h(24) = 3 – collision

look lower – occupied

look lower – empty

Exercise

Modify the put() method of the ATD below to implement open addressing with linear probing.

```
class Dictionary:
  def __init__(self, capacity):
    # each element will be a key/value pair
    self._pairs = [None] * capacity

  def _hash(self, k):
    return len(k) % len(self._pairs)

  def put(self, k, v):
    self._pairs[self._hash(k)] = [k,v]      #use the hash function

.....
```

Clusters

- *Cluster*: a sequence of adjacent, occupied entries in a hash table

- problems with open addressing with linear probing
  - colliding keys are inserted into empty locations below the collision location
  - on each collision, a key is added at the edge of a cluster
  - the edge of the cluster keeps growing
  - the edges begin to meet with other clusters
  - these combine to make *primary clusters*

Collision Resolution

open addressing

- idea: need a probe decrement that is *different* for keys that hash to the same value

simple example

- the use mod for the hash
- use quotient for the probe
  - note: cannot use 0
- probe decrement function p(key)

  the quotient of key after division by 7 (if the quotient is 0, then 1)

  or

  max(1, key / 7)

called *open addressing with double hashing*

Collision Resolution – double hashing

- functions

  h(key) = key % 7

  p(key) = max(1, key / 7)

- values for the keys: 10, 2, 19, 14, 24, 23

key	hash value	probe decrement
10	3	1
2	2	1
19	5	2
14	0	2
24	3	3
23	2	3

Collision Resolution – double hashing

key	hash value	probe decrement
10	3	1
2	2	1
19	5	2
14	0	2
24	3	3
23	2	3

hash table after inserting keys: 10, 2, 19, 14

0	1	2	3	4	5	6
14		2	10		19	

Collision Resolution – double hashing

key	hash value	probe decrement
10	3	1
2	2	1
19	5	2
14	0	2
24	3	3
23	2	3

Now insert key 24:

0	1	2	3	4	5	6
14		2	10		19	

Collision Resolution – double hashing

key	hash value	probe decrement
10	3	1
2	2	1
19	5	2
14	0	2
24	3	3
23	2	3

Now insert key 24:

0	1	2	3	4	5	6
14		2	10		19	

↑

h(24) = 3 collision

What is the decrement?
What is the probe sequence?

Collision Resolution – double hashing

key	hash value	probe decrement
10	3	1
2	2	1
19	5	2
14	0	2
24	3	3
23	2	3

Now insert key 24:

0	1	2	3	4	5	6
14		2	10	24	19	

h(24) = 3 collision

What is the decrement? 3

What is the probe sequence? 3, 0, 4

Exercise

key	hash value	probe decrement
10	3	1
2	2	1
19	5	2
14	0	2
24	3	3
23	2	3

Use double hashing to insert key 23:

0	1	2	3	4	5	6
14		2	10	24	19	

Collision Resolution

- open addressing with double hashing
  - compute the hash value
  - on collision, use the probe decrement function to determine what slot to visit next
  - wrap if necessary
- improvement over linear probing
  - when two keys collide, they usually follow different probe sequences when a search is made for an empty location
    - hash(10) = 3 hash(24) = 3
    - probe(10) = 1 probe(24) = 3
  - prevents primary clustering

Hash functions and collisions

- Consider an *ideal hash* function h(k)
  - it maps keys to hash values (slots) uniformly and randomly
- Suppose T is a hash table having M table entries from 0 to M-1
- An ideal hash function would imply that any slot from 0 to M -1 is equally likely
- All slots equally likely, implies collisions would be infrequent.
- Is that true?

collision phenomenon

- von Mises Birthday Paradox
  - if there are 23 or more people in a room, there is a > 50% chance that two or more will have the same birthday

collision phenomenon

Ball tossing model

Given

- a table T with 365 slots

  (each is a different day of the year)

- toss 23 balls at random into these 365 slots

then

- there is a > 50% chance we will toss 2 or more balls into the same slot

What?

- 23 balls in the table
- the table is only 6.3% full

  23/365 = .063

- and we have a 50% chance of a collision!

collision phenomenon

Ball tossing model

P(n) = probability that tossing n balls into 365 slots has at least one collision

$$P(n) = 1 - \frac{365!}{365^n (365-n)!}.$$

collision phenomenon

P(n) = probability that tossing n balls into 365 slots has at least one collision

n	P(n)
5	0.027
10	0.117
20	0.411
23	0.572
30	0.706
40	0.891
50	0.970
60	0.994
70	0.99915958
80	0.99991433
100	0.99999969

← at 23, greater than 50% chance

collision phenomenon

P(n) = probability that tossing n balls into 365 slots has at least one collision

n	P(n)
5	0.027
10	0.117
20	0.411
23	0.572
30	0.706
40	0.891
50	0.970
60	0.994
70	0.99915958
80	0.99991433
100	0.99999969

← at 23, greater than 50% chance

Our results:
58 people/ 365 possible birthdays
3 collisions:
- July 14
- Aug 1
- Aug 18

Collision resolution

A collision resolution algorithm must be guaranteed to check every slot.

linear probing - yes (it sequentially walks through the slots)

double hashing - ?

Does the probe sequence used for double hashing cover the entire table? (I.e., is any slot ever missed?)

Collision resolution – double hashing

key	hash value	probe decrement
10	3	1
2	2	1
19	5	2
14	0	2
24	3	3
23	2	3

Question: Does the probe sequence cover the entire table?

0	1	2	3	4	5	6

Use key 24. Show that the probe sequence visits each slot. (Keep wrapping.)

Collision resolution

The probe sequence covers every slot.

This is true for every key in the table

- *try it for other keys*

Why?

The table size M and probe decrement are *relatively prime*. Guarantees that the probe sequence covers the table.

relatively prime

- have no common divisors other than 1
- think of reducing the fraction 36/45 to 4/5

Collision resolution

Two policies

- open addressing
  - with linear probing
  - with double hashing

A third policy

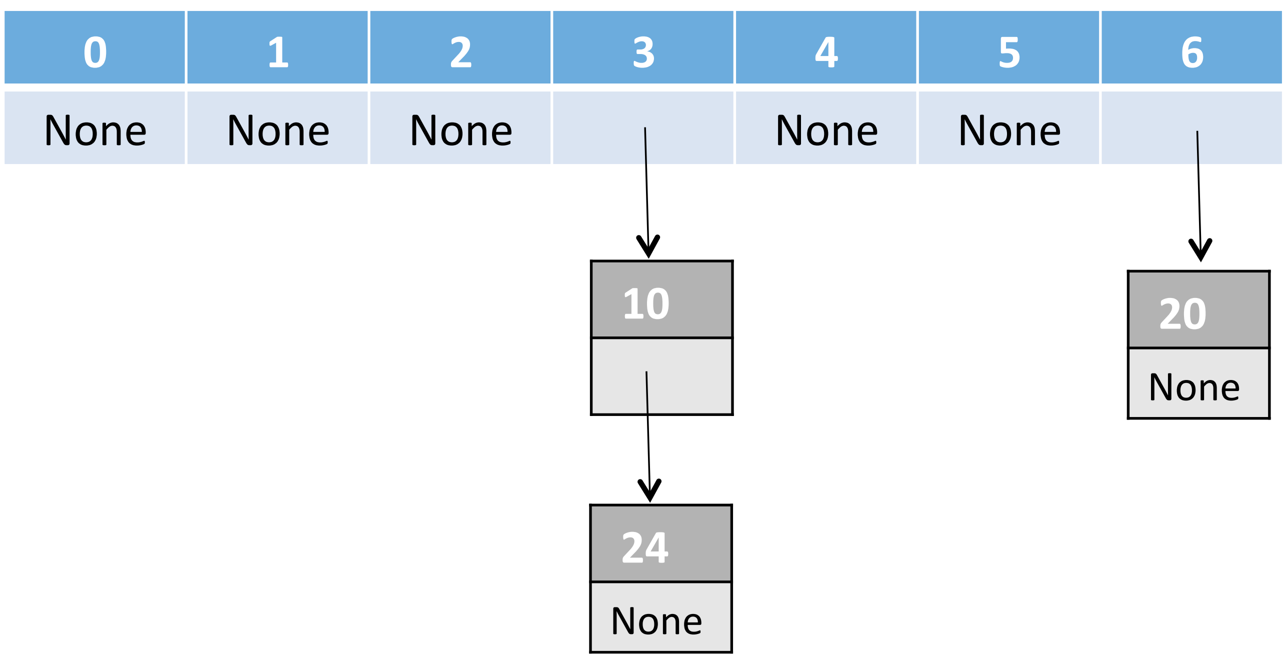
- separate chaining

Collision Resolution

separate chaining

- each table location references a linked list
- on collision, add to the linked list, starting at the collision slot

table with keys 24 and 10 (using %7 for the hash):



Complexity

Analysis of separate chaining

If we have N keys, what is

- best case complexity for search:
  (the key is the first item in the linked-list) O(1)
- worst case complexity for search:
  (must exhaustively search one linked-list) O(n)

We have not been analyzing the average case.

We will use known results for average case of the collision resolution policies.

Load factor

The load factor of a hash table with N keys and table size M is given by the following:

$$\lambda = N/M$$

load factor is a measure of how full the table is

Complexity is expressed in terms of the load factor.

EXERCISE

We have 60,000 items to store in a hash table using open addressing with linear probing and we want a load factor of .75.

How big should the hash table be?

Complexity

As load factor increases, efficiency of inserting new keys decreases

Collisions

- must enumerate through the table to get an empty slot

Searching

- find it on the first try
- search by using the probe sequence
- or search the linked list

We will use known results for the average cases of successful and unsuccessful search for the collision resolution policies

Assume a table with load factor: $\lambda = N/M$

Linear probing:

clusters form

leads to long probe sequences

It can be shown that the average number of probes is

$$\frac{1}{2}\left(1 + \frac{1}{1-\lambda}\right)$$ for successful search

$$\frac{1}{2}\left(1 + \frac{1}{(1-\lambda)^2}\right)$$ for unsuccessful search

Bad when load factor is close to 1

Not too bad when load factor is .75 or less

Results

```
>>> # load factor is .75
>>>
>>> # linear probing - successful
>>>
>>> .5 * (1 + 1/.25)
2.5
>>> # linear probing - unsuccessful
>>>
>>> .5 * ( 1 + 1/(.25 *.25))
8.5
```

Assume a table with load factor: $\lambda = N/M$

Double hashing:

clustering less common

It can be shown that the average number of probes is

$$\frac{1}{\lambda}\ln\left(\frac{1}{1-\lambda}\right)$$ for successful search

$$\left(\frac{1}{1-\lambda}\right)$$ for unsuccessful search

Very good when load factor is .75 or less

Results

```
>>> # load factor is .75
>>>
>>> # double hashing - successful
>>>
>>> import math
>>> 1/.75 * math.log(4)
1.8483924814931874
>>>
>>> # double hashing – unsuccessful
>>> 1/.25
4.0
```

Assume a table with load factor: $\lambda = N/M$

Separate chaining:

all keys that collide at a given has location are on the same linked list

It can be shown that the average number of probes is

$1 + \frac{1}{2}\lambda$ for successful search

λ for unsuccessful search

Compare the three methods

Theoretical Results (number of probes)

Successful search

Load Factor	0.50	0.75	0.90	0.99
separate chaining	1.25	1.37	1.45	1.49
linear probing	1.50	2.50	5.50	50.5
double hashing	1.39	1.85	2.56	4.65

Unsuccessful search

Load Factor	0.50	0.75	0.90	0.99
separate chaining	0.50	0.75	0.90	0.99
linear probing	2.50	8.50	50.50	5000.00
double hashing	2.00	4.00	10.00	100.00

Hashing Functions

Good performance requires a good hashing function.

- the hash function should not cause clustering

Most hash functions

- map keys to numbers (if not already numbers)
- then reduce that using mod

Example:

'hello' → len('hello') % 7

Must be aware of properties of the hashing function.

Hashing Functions

Example: hashing function *hash*

- add the ord values of a string
- mod by the table size M

For the key 'bat':

- hash('bat', M) = (ord('b') + ord('a') + ord('t')) % M

```
def hash(key, M):
    sum = 0
    for c in key:
        sum += ord(c)
    return sum % M
```

What are the properties of this hash function?

Does it cause clustering?

Hashing Functions

```
def hash(key, M):
    sum = 0
    for c in key:
        sum += ord(c)
    return sum % M
```

Use:

```
>>> hash("bat", 7)
3
>>> hash("tab", 7)
3
>>> hash("atb", 7)
3
>>> hash("tide", 7)
2
>>> hash("tied", 7)
2
```

Hashing Functions

Example: hashing function *h*

- add the ord values of a string
- mod by the table size M

hash('bat', M) = (ord('b') + ord('a') + ord('t')) % M

hash('tab', M) = (ord('t') + ord('a') + ord('b')) % M

What are the properties of this hash function?

- anagrams hash to the same value

Will that matter?

If it does, how would we fix that?

Hashing Functions

Example: hashing function *h*

- add the ord values of a string
- mod by the table size M

Modify to multiply by character position, i.e.,

hash('bat', M) = (ord('b')*1 + ord('a')*2 + ord('t')*3) % M

hash('tab', M) = (ord('t')*1 + ord('a')*2 + ord('b')*3) % M

Hashing Functions

Pitfalls with mod

$h(k) = k \bmod M$

Avoid powers of 2 for M

for $M = 2^{\mathbf{b}}$, $h(k) = k \bmod 2^b$

This elects the **b** low order bits of **k**

In general, when using mod

avoid powers of 2

use prime numbers for M