CSc 120

Introduction to Computer Programming II

08: Efficiency and Complexity


efficiency matters

reasoning about performance

Reasoning about efficiency

- Not *just* the time taken for a program to run
  - this can depend on:
    - processor properties that have nothing to do with the program *(e.g., CPU speed, amount of memory)*
    - what other programs are running *(i.e., system load)*
    - which inputs we use *(some inputs may be worse than others)*

- We would like to compare different algorithms:
  - without requiring that we implement them both first
  - abstracting away processor-specific details
  - considering all possible inputs

Primitive operations

- Abstract units of computation
  - convenient for reasoning about algorithms
  - approximates typical hardware-level operations
- Includes:
  - assigning a value to a variable
  - looking up the value of a variable
  - doing a single arithmetic operation
  - comparing two numbers
  - accessing a single element of a Python list by index
  - calling a function
  - returning from a function

Primitive ops and running time

- A primitive operation typically corresponds to a small constant number of machine instructions
- No. of primitive operations executed

  $\propto$ no. of machine instructions executed

  $\propto$ actual running time

- We consider how a function's running time depends on the size of its input
  - *which input do we consider?*

Best case vs. worst case inputs

*# lookup(str_, list_): returns the index where str_ occurs in list_*

```
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

- Best-case scenario: str_ == list_[0]   *# first element*
    - loop does not have to iterate over list_ at all
    - running time does not depend on length of list_
    - does not reflect typical behavior of the algorithm

Best case vs. worst case inputs

*# lookup(str_, list_): returns the index where str_ occurs in list_*

```
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

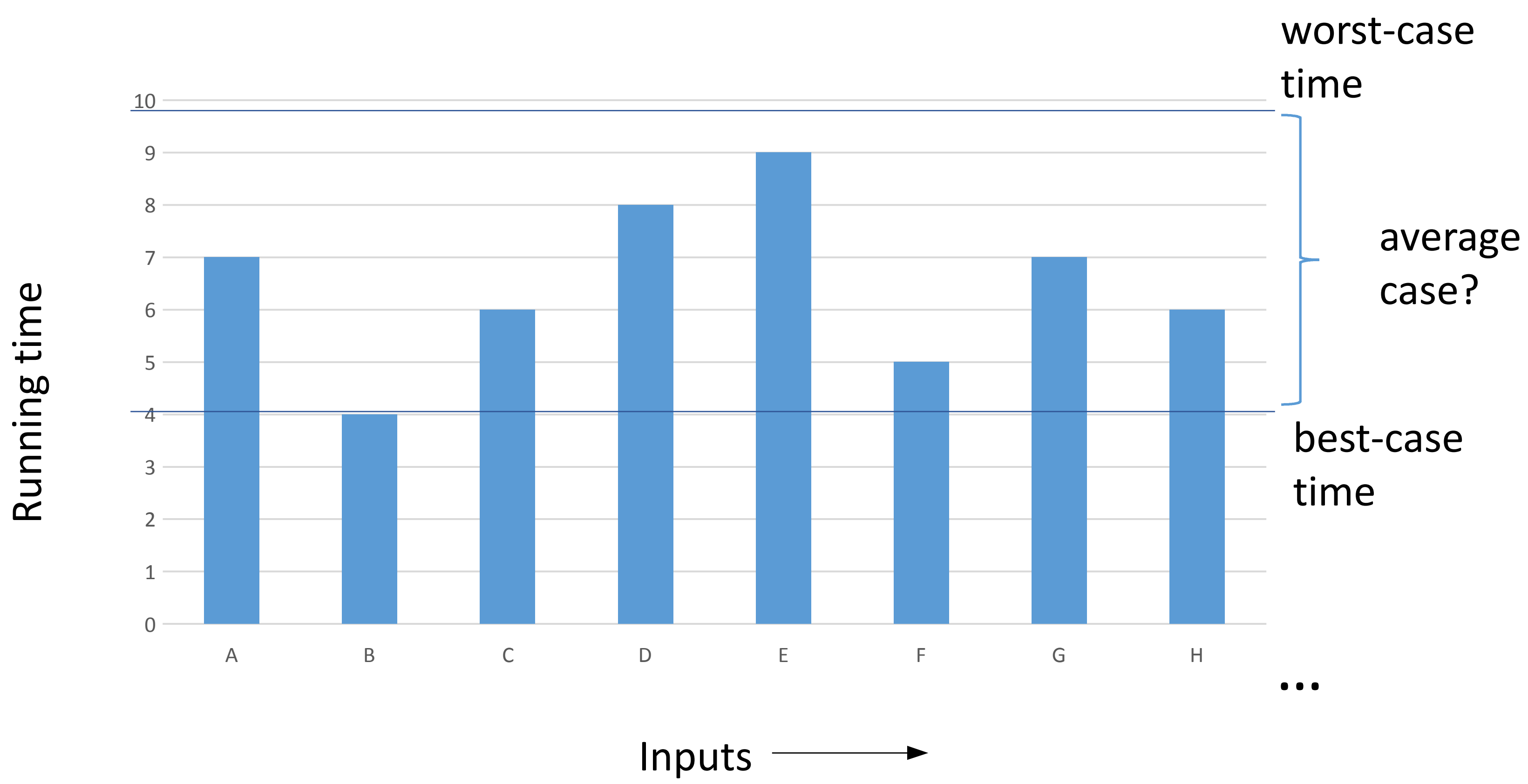
- Worst-case scenario: str_ == list_[-1]   *# last element*
  - loop iterates through list_
  - running time is proportional to the length of list_
  - captures the behavior of the algorithm better

Best case vs. worst case inputs

```
# lookup(str_, list_): returns the index where str_ occurs in list_
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

- In reality, we get something in between
  - but "average-case" is difficult to characterize precisely

What about “average case”?



Worst-case complexity

- Considers worst-case inputs
- Describes the running time of an algorithm as a function of the size of its input ("time complexity")
- Focuses on the *rate* at which the running time grows as the input gets large
- Typically gives a better characterization of an algorithm's performance

- This approach can also be applied to the amount of memory used by an algorithm ("space complexity")

Example

Code

```
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

Primitive operations

`for i in range(len(list_)):`

len(list_) :	1
range( ) :	1
in :	1
for :	2

`if str_ == list_[i]:`

list_[i] :	1
str_ :	1
== :	1
if :	1

each iteration:
9 primitive ops

Example

Code

```
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
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```

Primitive operations

for i in range(len(list_)):

len(list_) :	1
range( ) :	1
in :	1
for :	2

if str_ == list_[i]:

list_[i] :	1
str_ :	1
== :	1
if :	1

each iteration:
9 primitive ops

Total primitive ops executed:

1 iteration: 9 ops

∴ n iterations: 9n ops

+ return at the end: 1 op

∴ total worst-case running time for a list of length n = 9n + 1

asymptotic complexity

Asymptotic complexity

- In the worst-case, lookup(str_, list_) executes 9n + 1 primitive operations given a list of length n
- To translate this to running time:
  - suppose each primitive operation takes *k* time units
  - then worst-case running time is (9n + 1)*k*
- But *k* depends on specifics of the computer, e.g.:

Processor speed	*k*	running time
slow	20	180n + 20
medium	10	90n + 10
fast	3	27n + 3

Asymptotic complexity

depends on processor-specific characteristics

worst case running time = $An + B$

depends on how the algorithm processes data

Asymptotic complexity

- For algorithm analysis, we focus on how the running time grows as a function of the input size *n*
  - usually, we do not look at the <u>exact</u> worst case running time
  - it's enough to know proportionalities

- E.g., for the lookup() function:
  - we say only that its running time is *"proportional to the input length n"*

Example

Code

```
def list_positions(list1, list2):
    positions = []
    for value in list1:
        idx = lookup(value, list2)
        positions.append(idx)
    return positions
```

Example

Code	**Primitive operations**
`def list_positions(list1, list2):`	
`positions = []`	1
`for value in list1:`	in : 1 for : 2
`idx = lookup(value, list2)`	9n + 1
`positions.append(idx)`	1
`return positions`	1

(in, for, lookup and append: iterates n times)

Worst case behavior:

primitive operations $= n(9n + 5) + 2 = 9n^2 + 5n + 2$

running time $= k(9n^2 + 5n + 2)$

Example

Code

```
def list_positions(list1, list2):
    positions = []
    for value in list1:
        idx = lookup(value, list2)
        positions.append(idx)
    return positions
```

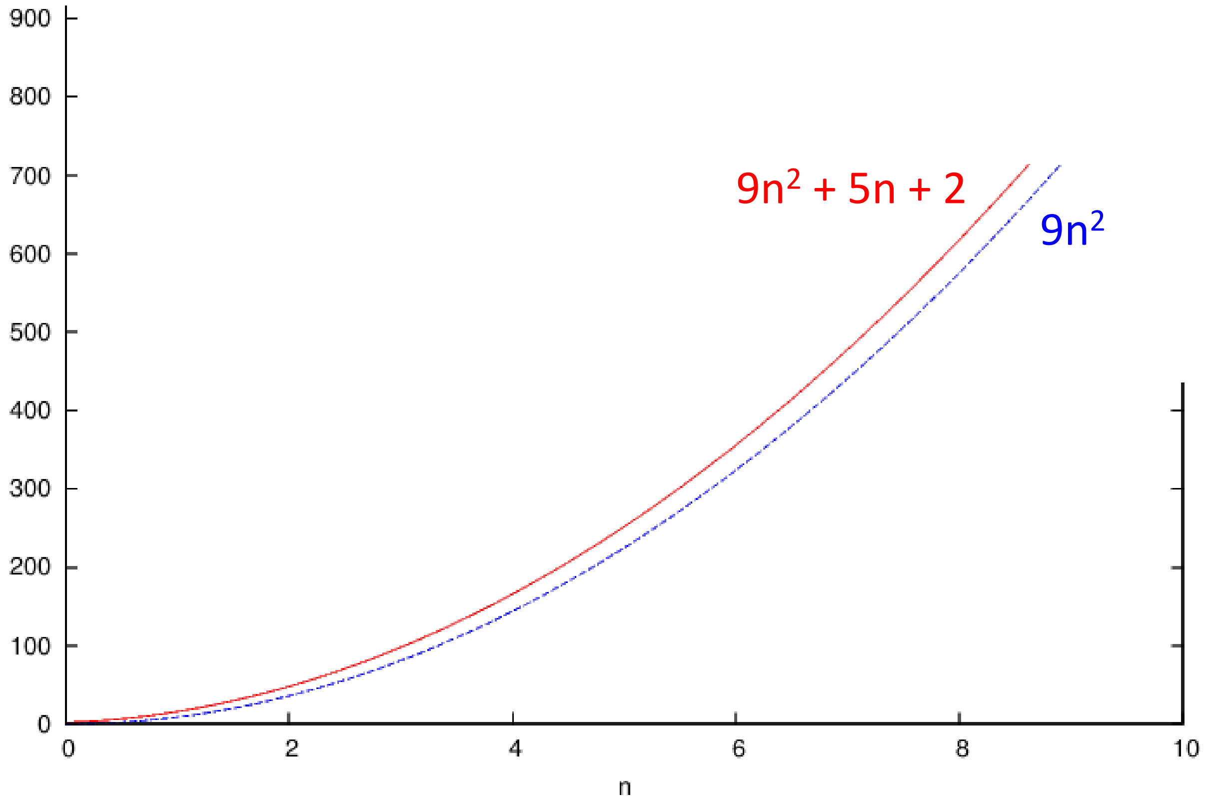
Worst case: $9n^2 + 5n + 2$

As n grows, the $9n^2$ term grows faster than 5n+2

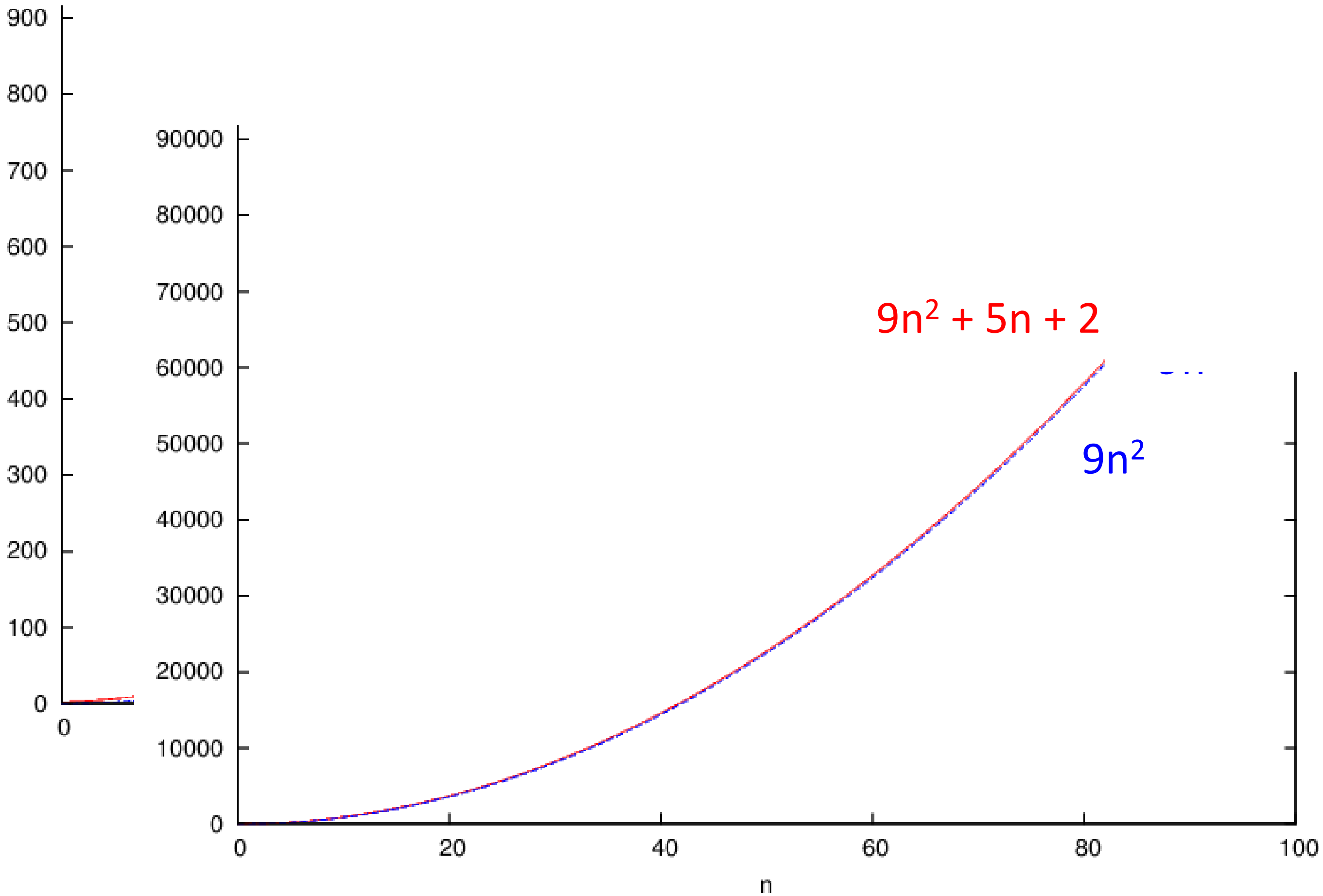
$\Rightarrow$ for large n, the n^2 term dominates

$\Rightarrow$ running time depends primarily on n^2

Example



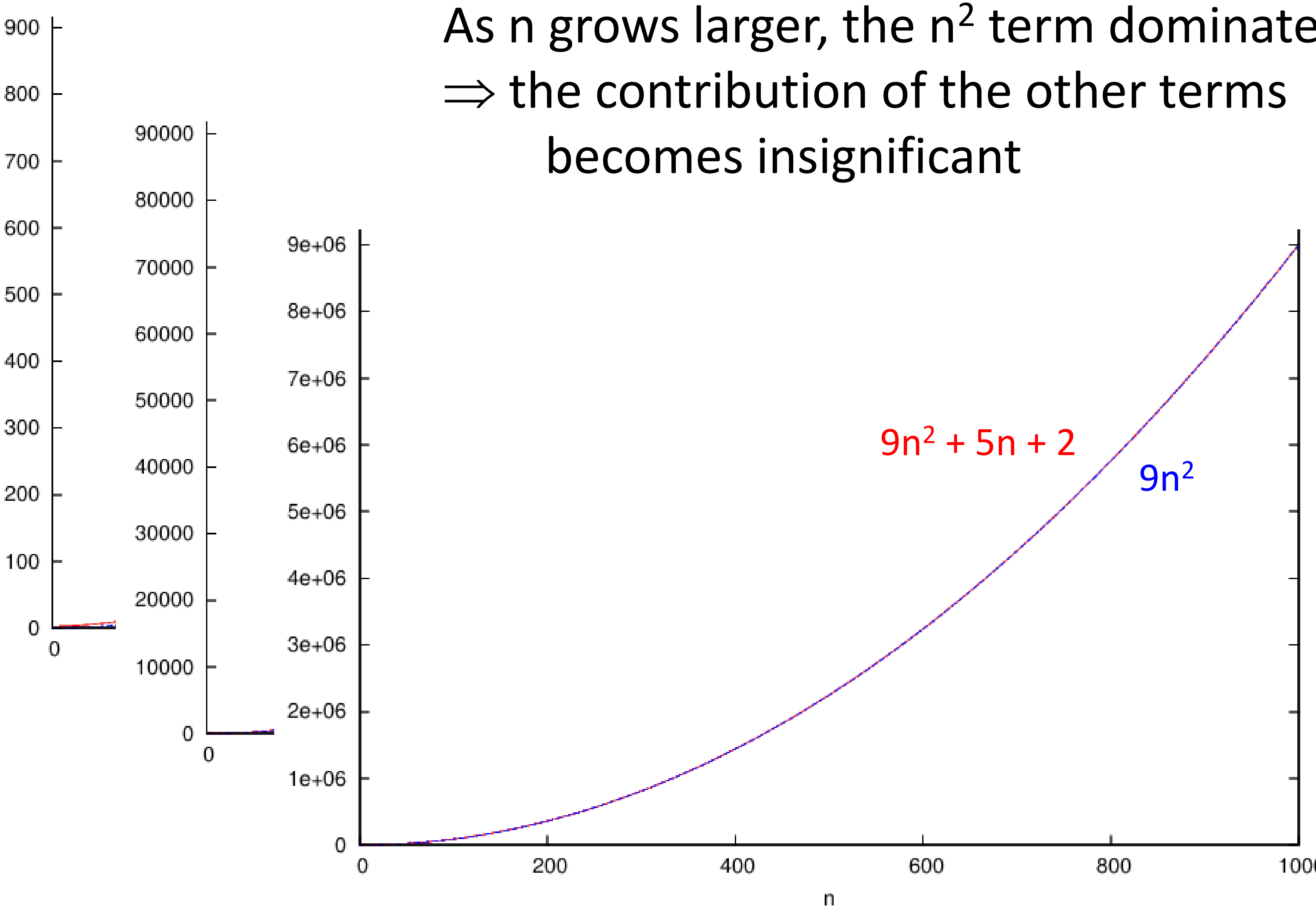
Example



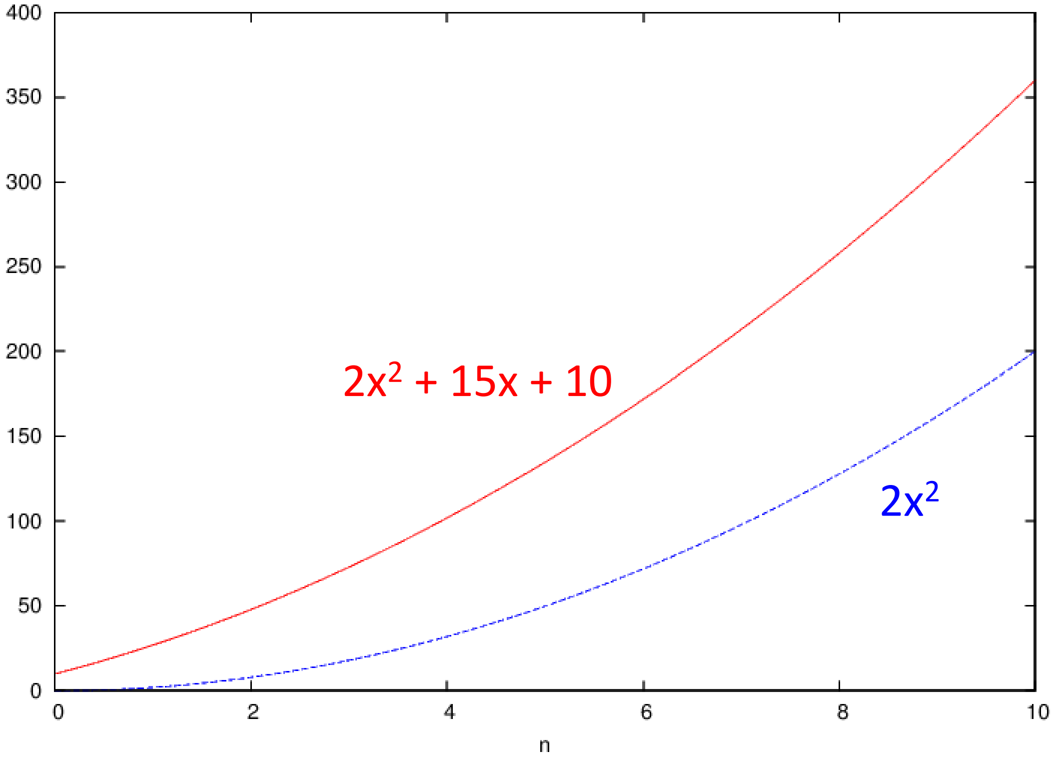
900
800
700
600
500
400
300
200
100
0
0
90000
80000
70000
60000
50000
40000
30000
20000
10000
0
0
20
40
60
80
100
n
9n² + 5n + 2
9n²

Example

As n grows larger, the n^2 term dominates
$\Rightarrow$ the contribution of the other terms becomes insignificant



Example 2: $2x^2 + 15x + 10$



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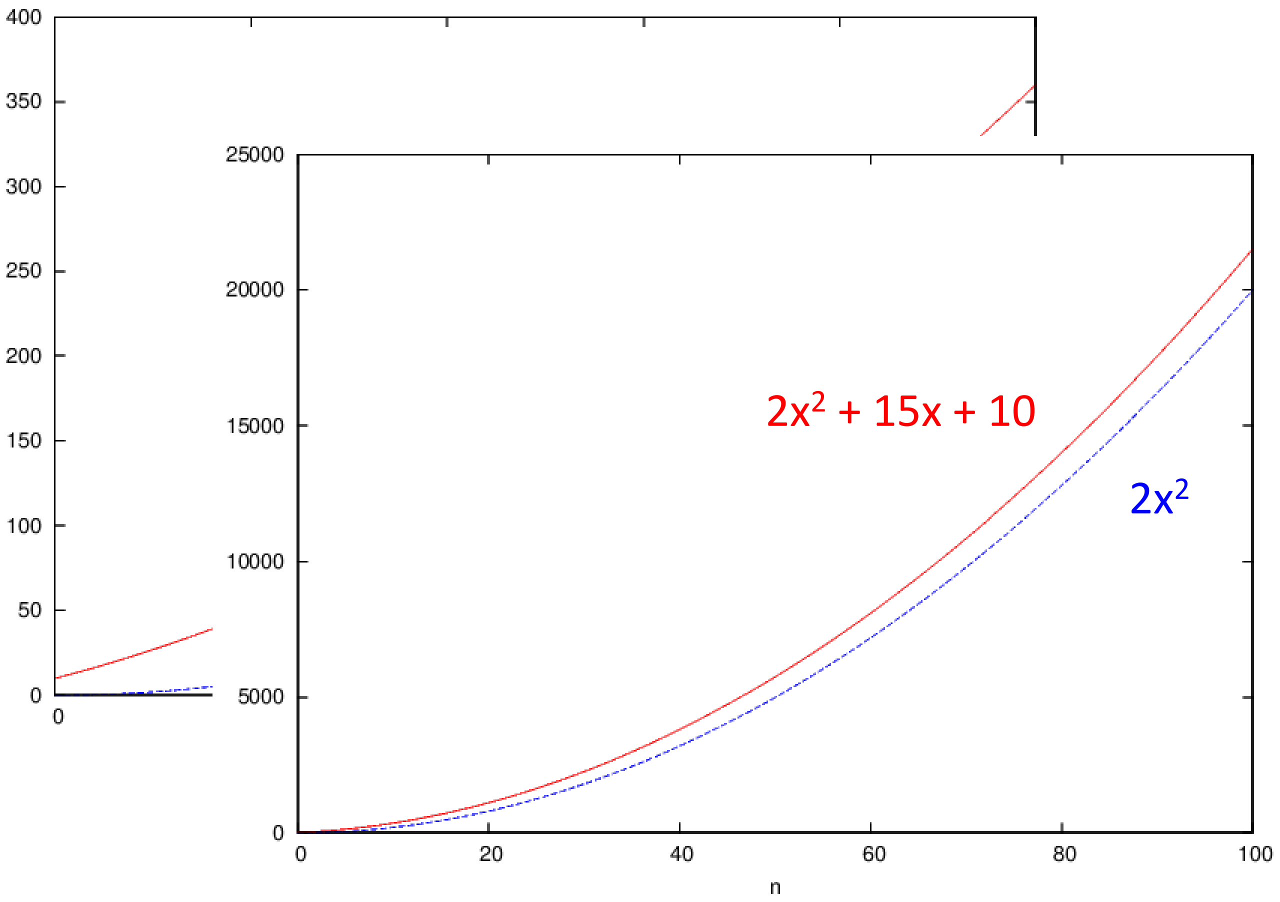
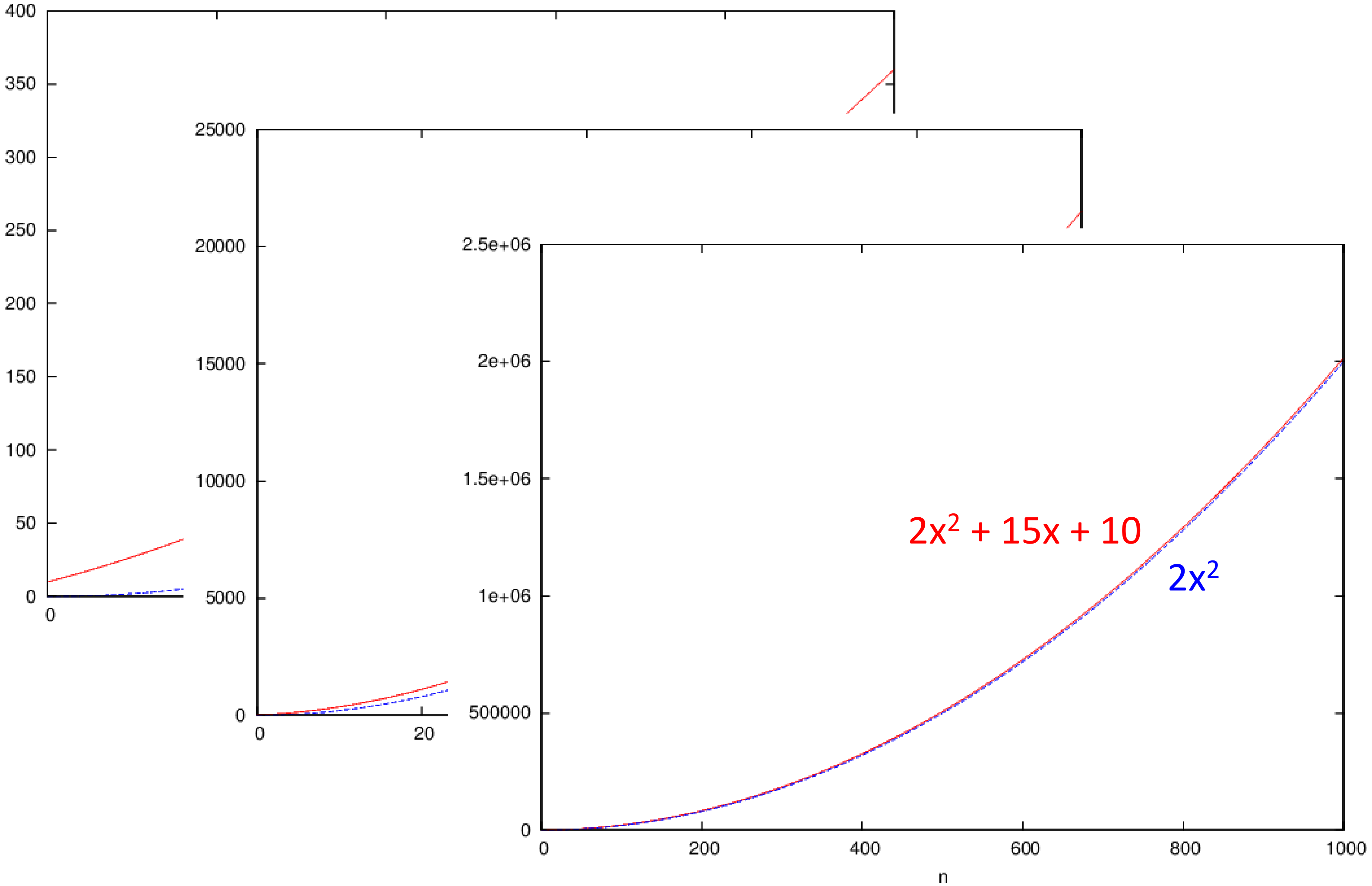
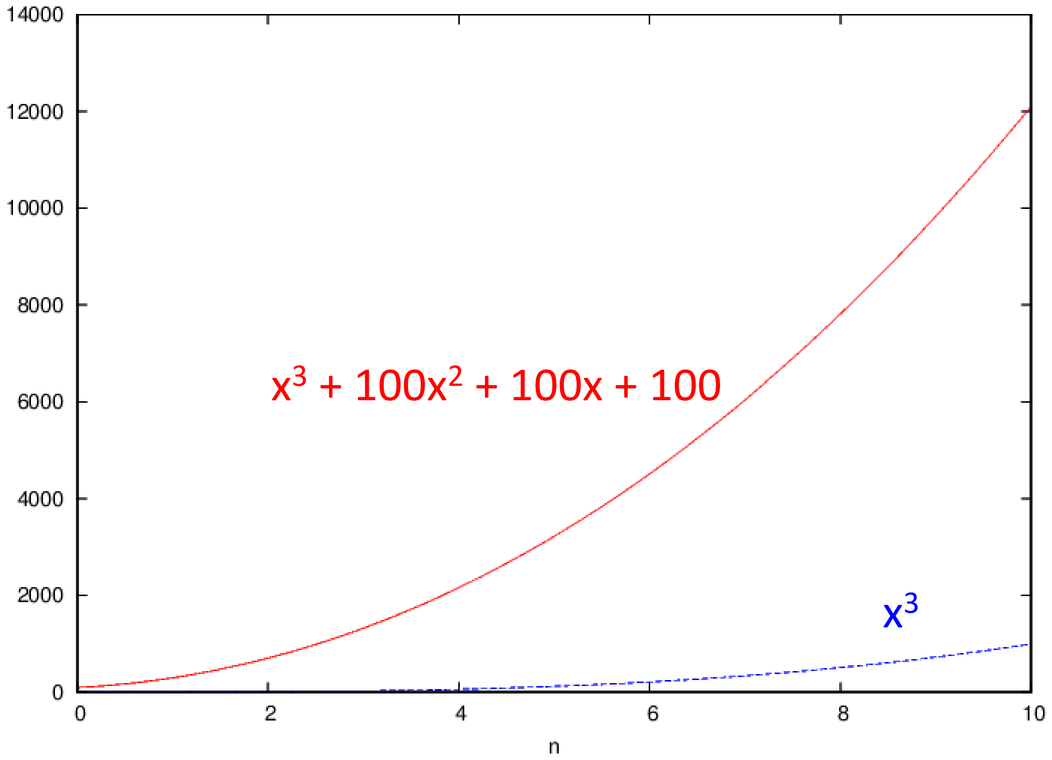


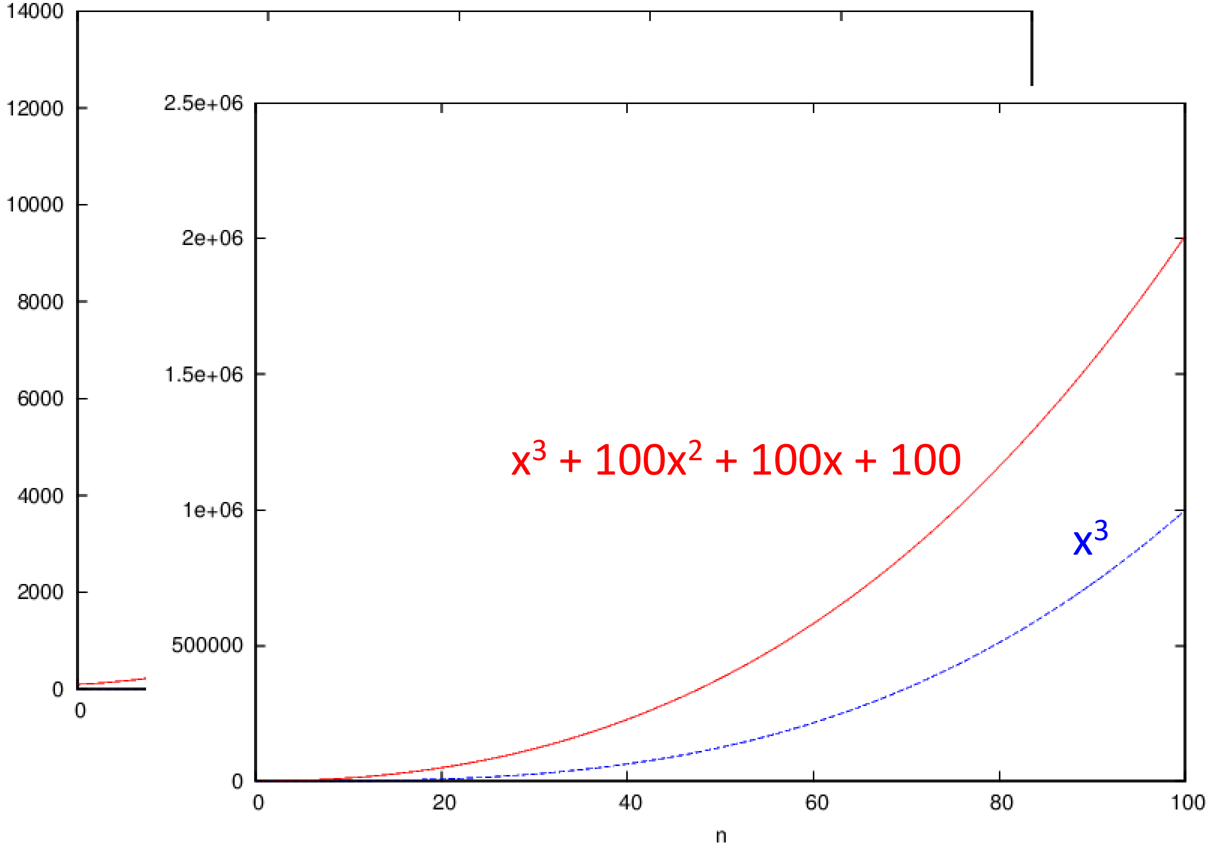
Example 2: $2x^2 + 15x + 10$



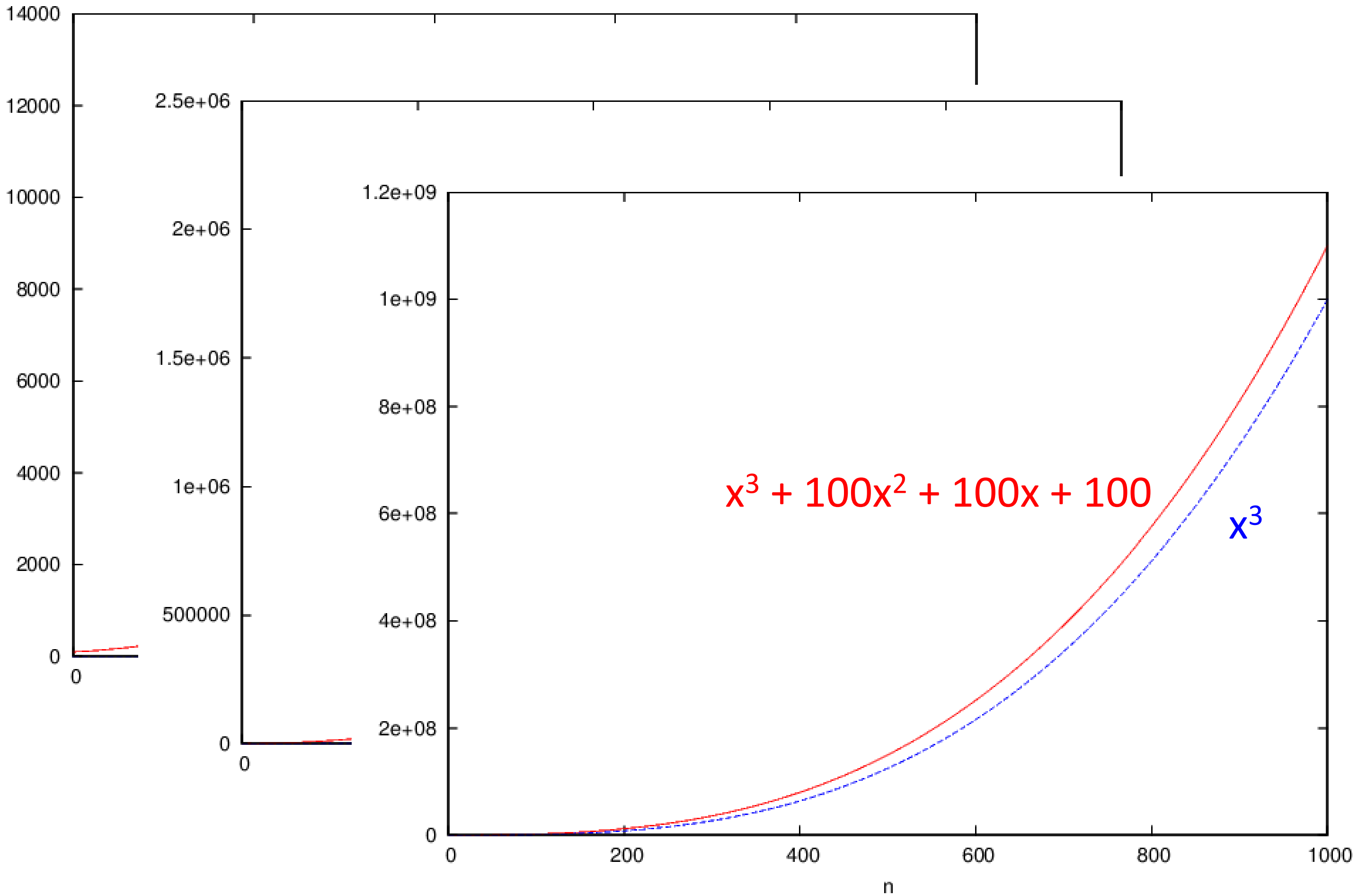
Example 3: $x^3 + 100x^2 + 100x + 100$



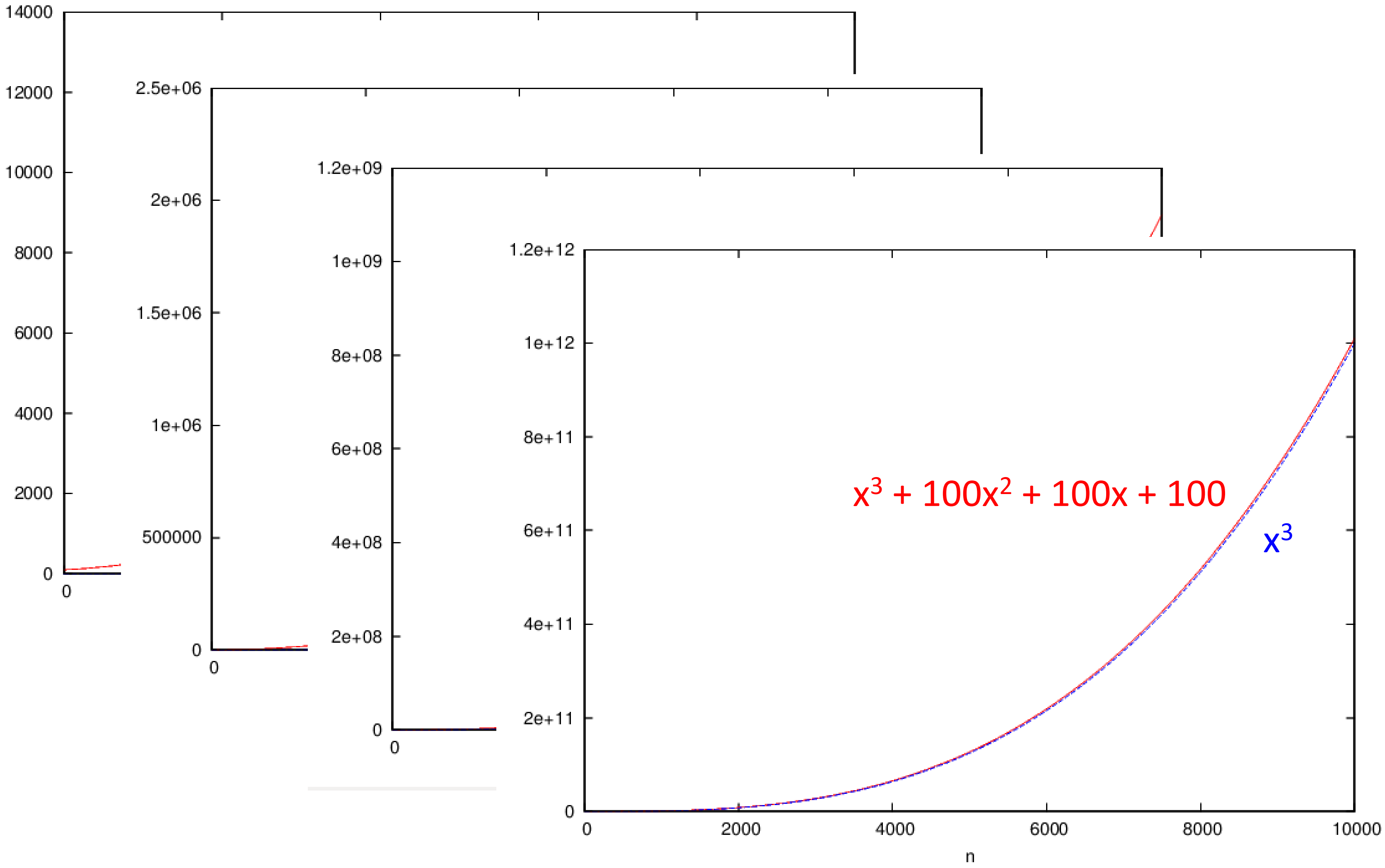
Example 3: $x^3 + 100x^2 + 100x + 100$



Example 3: $x^3 + 100x^2 + 100x + 100$



Example 3: $x^3 + 100x^2 + 100x + 100$



Growth rates

- As input size grows, the fastest-growing term dominates the others
  - the contribution of the smaller terms becomes negligible
  - it suffices to consider only the highest degree (i.e., fastest growing) term

- For algorithm analysis purposes, the constant factors are not useful
  - they usually reflect implementation-specific features
  - to compare different algorithms, we focus only on proportionality

  $\Rightarrow$ ignore constant coefficients

Comparing algorithms

Growth rate ∝ n

```
def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
```

Growth rate ∝ n^2

```
def list_positions(list1, list2):
    positions = []
    for value in list1:
        idx = lookup(value, list2)
        positions.append(idx)
    return positions
```

Summary so far

- Want to characterize algorithm efficiency such that:
  - does not depend on processor specifics
  - accounts for all possible inputs

  ⇒ count primitive operations

  ⇒ consider worst-case running time

- We specify the running time as a function of the size of the input
  - consider proportionality, ignore constant coefficients
  - consider only the dominant term
    - e.g., $9n^2 + 5n + 2 \approx n^2$

big-O notation

Big-O notation

Intuition:

When we say…	***…we mean***
"f(n) is O(g(n))"	"f is growing roughly as fast as g"

"big-O notation"

Big-O notation

- Captures the idea of the growth rate of functions, focusing on proportionality and ignoring constants

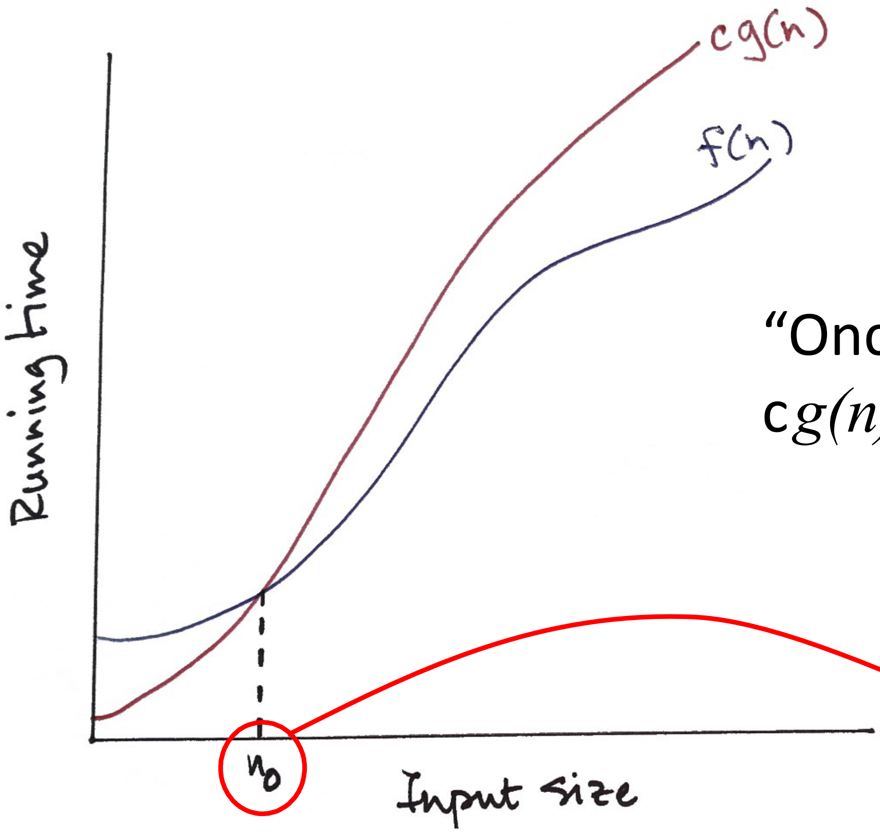
Definition: Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers.

Then, $f(n)$ is $O(g(n))$ if there is a real constant c and an integer constant $n_0 \geq 1$ such that

$$f(n) \leq c\,g(n) \quad \text{for all } n > n_0$$

Big-O notation

$f(n)$ is O($g(n)$) if there is a real constant c and an integer constant $n_0 \geq 1$ such that $f(n) \leq c\, g(n)$ for all $n > n_0$



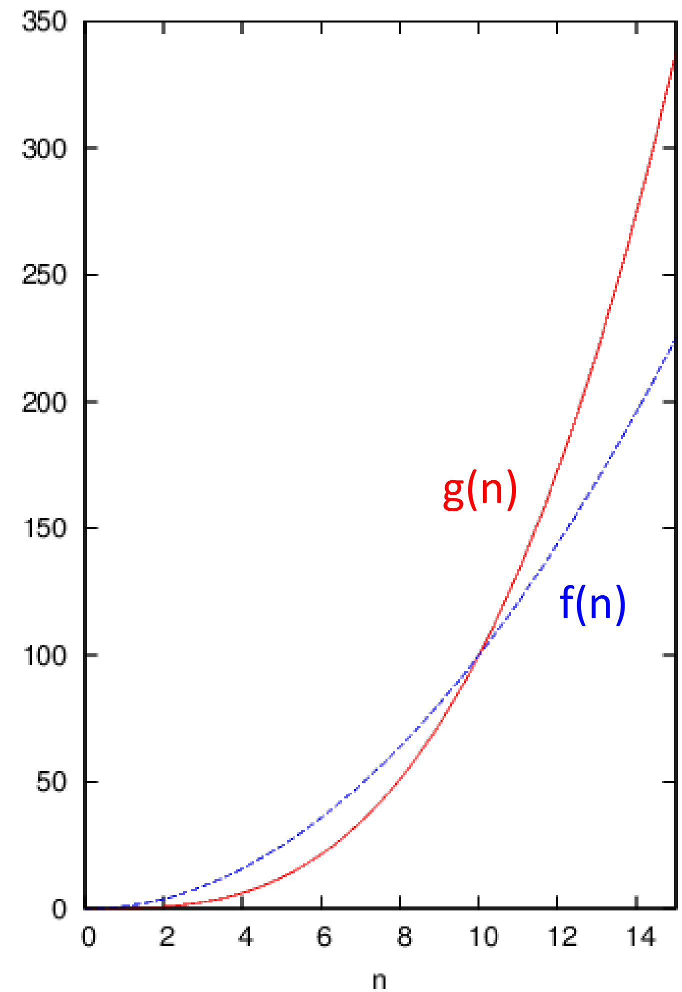
"Once the input gets big enough, c$g(n)$ is (always) larger than $f(n)$"

Big-O notation: properties

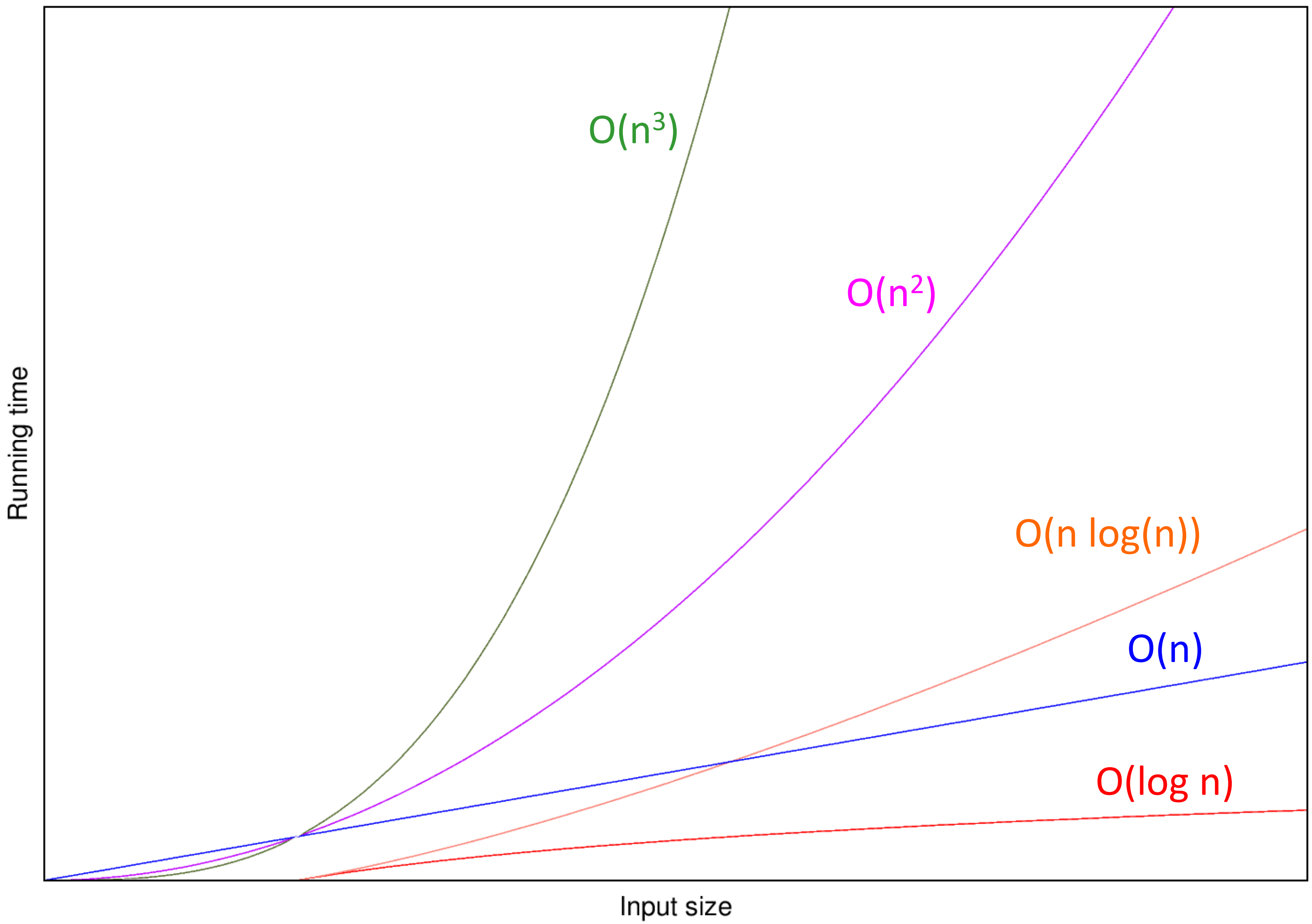
- If g(n) is growing faster than f(n):
  - f(n) is O(g(n))
  - g(n) is not O(f(n))
-
- If $f(n) = a_0 + a_1 n + \ldots + a_k n^k$, then:

$$f(n) = O(n^k)$$

  - i.e., coefficients and lower-order terms can be ignored



Some common growth-rate curves



using big-O notation

Using big-O notation

Code	Big-O complexity
O(1) str_ == list_[i] O(1) O(1)	O(1)

Using big-O notation

Code	Big-O complexity
`if str_ == list_[i]:` `    return i`	O(1)

O(1)

O(1)

O(1)

Using big-O notation

Code	Big-O complexity

`for i in range(len(list_)):`
`    if str_ == list_[i]:`
`        return i`
	O(n)

O(n) (worst-case)
(n = length of the list)

O(1)

Using big-O notation

Code	Big-O complexity

def lookup(str_, list_):
    for i in range(len(list_)):
        if str_ == list_[i]:
            return i
    return -1
	O(n)

O(n)

O(1)

Using big-O notation

Code	Big-O complexity
for value in list1: idx = lookup(value, list2)	$O(n^2)$

O(n) (worst-case)
(n = length of list1)

O(n) (worst-case)
(n = length of list2)

Using big-O notation

Code	Big-O complexity

```
def list_positions(list1, list2):
    positions = []
    for value in list1:
        idx = lookup(value, list2)
        positions.append(idx)
    return positions
```

$O(1)$ (annotation on `positions = []` and `return positions`)

$O(n^2)$ (annotation on the `for` loop)

Big-O complexity: $O(n^2)$

Computing big-O complexities

Given the code:

$line_1$ ... $O(f_1(n))$

$line_2$ ... $O(f_2(n))$

...

$line_k$ ... $O(f_k(n))$

The overall complexity is

$O(max(f_1(n), f_s(n), ..., f_k(n)))$

Given the code

loop ... O(f1(n)) iterations

    line1 ... O(f2(n))

The overall complexity is

$O(f_1(n) \times f_2(n))$

EXERCISE

```
# my_rfind(mylist, elt) : find the distance from the
# end of mylist where elt occurs, -1 if it does not
def my_rfind(mylist, elt):
    pos = len(mylist) – 1
    while pos >= 0:
        if mylist[pos] == elt:
            return pos
        pos -= 1
    return -1
```

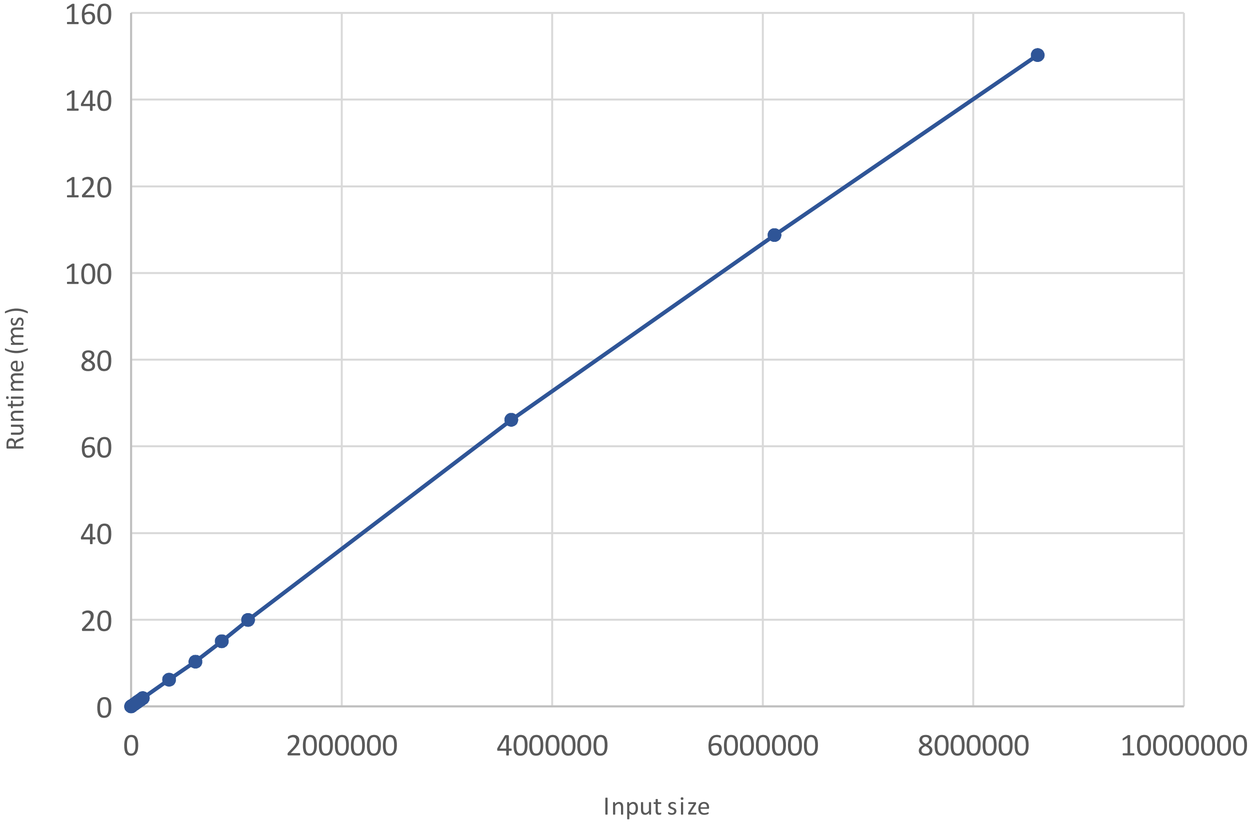
Worst-case big-O complexity = ???

EXERCISE

```
# for each element of a list: find the biggest value
# between that element and the end of the list
def find_biggest_after(arglist):
    pos_list = []
    for idx0 in range(len(arglist)):
        biggest = arglist[idx0]
        for idx1 in range(idx0+1, len(arglist)):
            biggest = max(arglist[idx1], biggest)
        pos_list.append(biggest)
    return pos_list
```

Worst-case big-O complexity = ???

Input size vs. run time: max()



EXERCISE

```
# for each element of a list: find the biggest value
# between that element and the end of the list
def find_biggest_after(arglist):
    pos_list = []
    for idx0 in range(len(arglist)):
        biggest = max(arglist[idx0:])   # library code
        pos_list.append(biggest)
    return pos_list
```

Worst-case big-O complexity = ???