Topic 3:

Quantification

Propositions With Variables (1 / 2)

Propositions are static; variables are not allowed. But . . .

Definition: Predicate (a.k.a. Propositional Function)

. . . .

. . . .

Example(s):

Propositions With Variables (2 / 2)

Definition: Domain (a.k.a. Universe) of Discourse

. . . .

Example(s):

Quantification

Idea: Establish truth of predicates over sets of values.

Two common generalizations:

Note: Do not use the book's non-standard $\exists!x$ notation.

Evaluating Quantified Predicates (1 / 2)

1. Universally Quantified Predicates

Example(s):

Evaluating Quantified Predicates (2 / 2)

2. Existentially Quantified Predicates

Example(s):

Evaluating Mixed Quantifications (1 / 2)

First: Distinguishing $\exists x\, \forall y\, S(x,y)$ from $\forall i\, \exists k\, T(i,k)$:

Evaluating Mixed Quantifications (2 / 2)

Example(s):

Example: Universal Quantification (1 / 5)

Consider this conversational English statement:

All of McCann's students are geniuses.

How can we express that statement in logic notation?

Example: Universal Quantification (2 / 5)

Attempt #2: All of McCann's students are geniuses. $\longrightarrow$ Logic

Example: Universal Quantification (3 / 5)

Attempt #3: All of McCann's students are geniuses. $\longrightarrow$ Logic

Let $P(x)$: Student x is a genius, $x \in$ People

Example: Universal Quantification (4 / 5)

Attempt #4: All of McCann's students are geniuses. $\longrightarrow$ Logic

Let $P(x)$: Student x is a genius, $x \in$ People

Let $M(x)$: x is enrolled in one of McCann's classes, $x \in$ People

Example: Universal Quantification (5 / 5)

Attempt #5: All of McCann's students are geniuses. $\longrightarrow$ Logic

Let $P(x)$: Student x is a genius, $x \in$ People

Let $M(x)$: x is enrolled in one of McCann's classes, $x \in$ People

Implicit Quantification

The "all" can be implicit in the English statement.

Example(s):

Example: Existential Quantification

Consider this conversational English statement:

At least one towel is dirty.

How can we express that statement in logic notation?

Another Example: Existential Quantification

Express this more specific statement in logic:

Some of the blue guest towels are dirty.

Let $D(x)$: x is dirty, $x \in$ Towels

Yet Another Example: Quantification

Now express this statement in logic:

Every last one of the blue guest towels are dirty!

Let $B(x)$: x is blue, $x \in$ Towels

Let $G(x)$: x is used by guests, $x \in$ Towels

Let $D(x)$: x is dirty, $x \in$ Towels

Free vs. Bound Variables

Definition: Bound Variable

Definition: Free (a.k.a. Unbound) Variable

Other examples of 'binding' in CS:

Negations of Quantified Expressions

Remember De Morgan's Laws for propositions? Well, . . .

Definition: Generalized De Morgan's Laws

. . . .

. . . .

. . . .

. . . .

Demonstration: $\overline{\forall x\, P(x)} \;\equiv\; \exists x\, \overline{P(x)}$ (1 / 2)

Let $S(x):\ x < 4, x \in \mathbb{Z}$

The expression $\forall x\ S(x), x \in \{1, 2, 3\}$ **is true.**

Equivalently, $\overline{\forall x\ S(x)}$ **is false.**

$\forall x\ S(x) \;\equiv\; S(1) \wedge S(2) \wedge S(3)$ **so . . .**

$$\overline{\forall x\ S(x)} \;\equiv\; \overline{S(1) \wedge S(2) \wedge S(3)}$$

$$\equiv\; \overline{S(1)} \vee \overline{S(2)} \vee \overline{S(3)}$$ **[De Morgan, 2x]**

(Remember: $\overline{S(1)} \vee \overline{S(2)} \vee \overline{S(3)}$ **is still false.)**

(Continues . . .)

Demonstration: $\overline{\forall x\, P(x)} \equiv \exists x\, \overline{P(x)}$ (2 / 2)

For $\overline{S(1)} \vee \overline{S(2)} \vee \overline{S(3)}$ to be false, each term must be false; that is, no $\overline{S(x)}$ is true (or all $\overline{S(x)}$ are false).

It follows that the expression $\exists x\, \overline{S(x)}$ must be false, completing the demonstration.

Example(s):

Expressing "Exactly one ..." Statements (1 / 3)

Consider this conversational (& correct!) English statement:

> Only one citizen of North Dakota is a member of the U.S. House of Representatives.

And consider this awkward but useful rewording:

Expressing “Exactly one . . . ” Statements (2 / 3)

That rewording is useful because it can be directly expressed logically:

Expressing “Exactly one . . . ” Statements (3 / 3)

A lingering problem:

The domain (“Citizens of North Dakota”) is too specific.

Solution: Add a predicate . . . but what, and where?

Expressing "Exactly two ..." Statements (1 / 3)

Key observation:

Question: Can you say this in 'awkward English'?

Exactly two citizens of North Dakota are U.S. Senators.

Expressing "Exactly two ..." Statements (2 / 3)

Consider the two halves separately. Given:

$S(x)$: x is a U.S. Senator, $x \in$ People

1. "At least two citizens of North Dakota are U.S. Senators"

2. "At most two citizens of North Dakota are U.S. Senators"

Expressing "Exactly two . . ." Statements (3 / 3)

Finally, AND together

$$\exists x\, \exists y\, (S(x) \wedge S(y) \wedge (x \neq y))$$

and

$$\forall x\, \forall y\, \forall z\, ((S(x) \wedge S(y) \wedge S(z)) \rightarrow (x = y \vee y = z \vee x = z)):$$