Topic 12:

Methods of Counting

The first math class.

Credit: www.smbc-comics.com/comic/a-new-method

The Pigeonhole Principle (1 / 2) (a.k.a. The Dirichlet Drawer Principle)

Example:

Definition: Pigeonhole Principle

. . . .

Definition: Pigeonhole Principle (w/ functions)

. . . .

The Pigeonhole Principle (2 / 2)

Example(s):

The Multiplication Principle (1 / 2)

Example(s):

Definition: Multiplication Principle (a.k.a. Product Rule)

. . . .

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The Multiplication Principle (2 / 2)

Example(s):

The Addition Principle (1 / 2)

Definition: Addition Principle (a.k.a. Sum Rule)

. . . .

. . . .

Example(s):

The Addition Principle (2 / 2)

Example(s):

The Principle of Inclusion-Exclusion (1 / 5)

A problem with the Addition Principle:

Example(s):

The Principle of Inclusion-Exclusion (2 / 5)

Definition: Principle of Inclusion-Exclusion for Two Sets

The Principle of Inclusion-Exclusion (3 / 5)

Definition: Principle of Inclusion-Exclusion for Three Sets

The cardinality of the union of sets M, N, and O is the sum of their individual cardinalities excluding the sum of the cardinalities of their pairwise intersections but including the cardinality of their intersection.

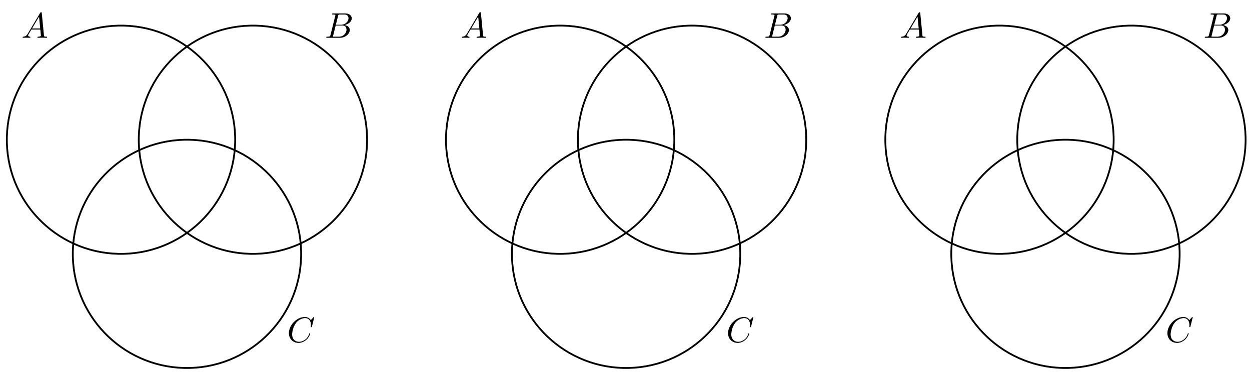
That is: $|M \cup N \cup O| = |M| + |N| + |O|$

$$-(|M \cap N| + |M \cap O| + |N \cap O|)$$

$$+|M \cap N \cap O|.$$

The Principle of Inclusion-Exclusion (4 / 5)

Why so complex?



The Principle of Inclusion-Exclusion (5 / 5)

Example(s):

Permutations (1 / 2)

Definition: Permutation

Example(s):

Permutations (2 / 2)

Conjecture: **There are $n!$ possible permutations of n elements.**

Example(s):

r-Permutations (1 / 3)

Definition: r-Permutation

. . . .

Conjecture: The number of r-permutations of n elements, denoted $P(n, r)$, is $n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$, $r \leq n$.

r-Permutations (2 / 3)

Observation:

Example(s):

r-Permutations (3 / 3)

Example(s):

r-Combinations (1 / 3)

Definition: r-Combination

. . . .

Other Notations:

Example(s):

r-Combinations (2 / 3)

The r-Permutation – r-Combination Connection:

Example(s):

r-Combinations (3 / 3)

Example(s):

Repetition and Permutations

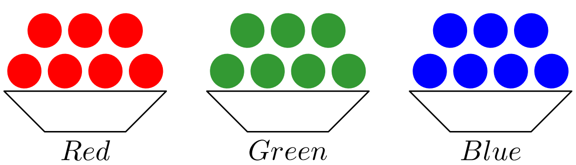
We've already seen this!

Example(s):

In General: When object repetition is permitted, the number of r-permutations of a set of n objects is $\boxed{n^r}$.

Repetition and Combinations (1 / 3)

Example(s): ‘Experienced’ Golf Balls



Repetition and Combinations (2 / 3)

Example(s):

In General: When repetition is allowed, the number of r–combinations of a set of n elements is $\boxed{\binom{n+r-1}{r} = \binom{n+r-1}{n-1}}$.

Repetition and Combinations (3 / 3)

A Small Extension:

Example(s):

In General: When repetition is allowed, the number of r–combinations of a set of n elements when one of each element is included in r is $\boxed{\binom{r-1}{r-n} = \binom{r-1}{n-1}}$.

Another View of Repetition and Combinations (1 / 2)

Consider: An integer variable can represent the quantity of items selected with repetition.

Example(s):

Another View of Repetition and Combinations (2 / 2)

Example(s):

Generalized Permutations (1 / 3)

Idea: What if some elements are indistinguishable?

Example(s):

Generalized Permutations (2 / 3)

What if we have indistinguishable copies of multiple elements?

Example(s):

In General: If we have n objects of t different types, and there are i_k indistinguishable objects of type k, then the number of distinct arrangements is $\boxed{P(n; i_1, i_2, \ldots, i_t) = \frac{n!}{i_1! \cdot i_2! \cdot \ldots \cdot i_t!}}$.

Generalized Permutations (3 / 3)

We can view $P(n; i_1, i_2, \ldots, i_t)$ in terms of combinations:

Example(s):

In General:

$$P(n; i_1, i_2, \ldots, i_t) = \binom{n}{i_1}\binom{n-i_1}{i_2}\binom{n-i_1-i_2}{i_3}\cdots\binom{n-\ldots-i_{t-1}}{i_t}$$

More Fun with Combinations (1 / 2)

What if we created a table of $\binom{n}{k}$ values?

k

n \ k	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

More Fun with Combinations (2 / 2)

Pascal's Triangle is the centered rows of the $\binom{n}{k}$ table:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

Proving that Pascal's Triange is 'Palindromic'

Conjecture: $\binom{n}{k} = \binom{n}{n-k}$**, where** $0 \leq k \leq n$

Pascal's Identity (Combinatorial Argument Example)

Conjecture: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, **where** $1 \leq k \leq n$

Pascal's Identity [Combinatorial Proof (1 / 2)]

Definition: Combinatorial Proof

Conjecture: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, **where** $1 \leq k \leq n$

Pascal's Identity [Combinatorial Proof (2 / 2)]

The Binomial Theorem (1 / 2)

The values of Pascal's Triangle appear in numerous places.

For instance:

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

Generalize this, and you've got the Binomial Theorem.

The Binomial Theorem (2 / 2)

Theorem: $(a+b)^n = \sum_{k=0}^{n} \left[\binom{n}{k} \cdot a^{n-k} \cdot b^k \right]$

Example(s):