Demonstrating Logical Equivalences Without Truth Tables

Truth tables work for demonstrating tautologies, contradictions, and other logical equivalences, but they get unwieldy when there are lots of statement labels and/or many operators. Instead, we can demonstrate that two propositions are equivalent by using a sequence of equivalences. For example, if we can show that $a \equiv b$ and that $b \equiv c$, then we know that $a \equiv c$ by transitivity of logical equivalence. Another possible approach is to use the power of our mind (informed by logic, of course!) to reason our way through the problem. This handout demonstrates both approaches on the same expression.

Example #1: Show that $r \rightarrow s \equiv \neg s \rightarrow \neg r$, using a sequence of logical equivalences.

Yes, this is just contrapositive (a.k.a. the Law of Contraposition), which we showed in class as being an equivalence. And, it's on the POLE (Page O' Logical Equivalences) handout (available from the class web page) on the second line of Table III. We're going to ignore all of that, and show the equivalence using other parts of that handout.

Here's how: By the Law of Implication (Table III, Line a), $r \rightarrow s \equiv \neg r \vee s$. But now what? Consider where we want to end up; we need s on the left of an implication operator and r on its right. We can swap sides with Commutativity (Table I(d)), giving $\neg r \vee s \equiv s \vee \neg r$. This is good, at least on the right side – we need a $\neg r$, and we have it. But we need to turn the OR back into an implication. The Law of Implication does that, but the form isn't quite right yet – we need the s to have a negation, so that it can be dropped as we move from the form $\neg p \vee q$ to the form $p \rightarrow q$. Actually, we need two negations, so that we can drop one and have one left. Ah! Double Negation lets us add or remove pairs of negations! That is, $s \vee \neg r \equiv \neg \neg s \vee \neg r$. Now we can apply the Law of Implication again: $\neg \neg s \vee \neg r \equiv \neg s \rightarrow \neg r$, and we're done.

Because that long narrative is as big a pain to read as it is to write, we prefer a tabular format. Here's the same sequence of equivalences in a much easier-to-read form:

$$
\begin{array}{rcll}
r \rightarrow s & \equiv & \neg r \vee s & \text{[Law of Implication]} \\
& \equiv & s \vee \neg r & \text{[Commutativity of } \vee \text{]} \\
& \equiv & \neg \neg s \vee \neg r & \text{[Double Negation]} \\
& \equiv & \neg s \rightarrow \neg r & \text{[Law of Implication]}
\end{array}
$$

Note that each equivalence used has an explanation; this is critical to the easy understanding of the sequence. The reader needs to be able to verify that each equivalence is applied correctly, and providing the names of them (or, if they don't have names, the table and line letters) makes verification much easier.

Now you know what we're looking to see. Want to see more examples? Look in section 1.2 of the Rosen text, and/or in "Kneel Before Zodd."

Be aware that, sooner or later, you will work a problem and will start down one path of equivalences only to hit a dead end — a position from which you can't see a way to move forward. It might be that you aren't looking at both sides of the available POLE equivalences (they can be used left–to–right or right–to–left), or it might be that you need to back up and try a different equivalence earlier on. Because of these "dead ends," these problems can sometimes be a little frustrating. Practice and patience help a lot!

That was one approach; let's show the same equivalence, this time using reasoning!

Example #2: Show that $r \to s \equiv \neg s \to \neg r$, this time using reasoning.

By "reasoning," we mean "without using a truth table, and without using a single sequence of logical equivalences." We'll still use our knowledge of those things, just in a different format.

The usual starting point is this observation: Each proposition letter in the expression is either true or false. Thus, if we can show that our expression is an equivalence when one of the letters is true, and also when that same letter is false, then it must always be an equivalence, as there are no other options to consider.

The first step is to decide which letter in the expression to use. Often, the best (easiest!) choice is the letter that appears most often in the expression. In this expression, neither r nor s is most frequent, so we'll just pick one: r.

$\boxed{\text{When } r \text{ is true:}}$ We have need to show that $T \to s \equiv \neg s \to \neg T$ is an equivalence. Let's start with the left side. From the truth table for implication, we know that true implying something is equivalent to that something ($T \to F \equiv F$, and $T \to T \equiv T$). Thus, $T \to s \equiv s$. If the right–hand side is also equivalent to s, we'll have established that the original expression is an equivalence when r is true. $\neg T \equiv F$, of course, and we know (from implication's truth table) that anything implies false is equivalent to the negation of the 'anything.' (Recall: $T \to F \equiv F$ and $F \to F \equiv T$.) In this case, our 'anything' is $\neg s$, and the negation of $\neg s$ is $\neg \neg s$, which is s via double negation. We've shown that both sides are equivalent to s, and s is certainly equivalent to itself.

Whew! Are we done? Sorry, no. That was just the reasoning through the case that r was true. We still have to show that both sides are equivalent when r is false. Hopefully that will be easier. One way to find out!

$\boxed{\text{When } r \text{ is false:}}$ We need to show that $F \to s \equiv \neg s \to \neg F$ is an equivalence. Starting with the left side, we know that $F \to$ anything is true, again from implication's truth table. That was easy! On the right side, we have something implies T. Is "something implies T" always true? It is, because $T \to T \equiv T$ and $F \to T \equiv T$. So, when r is false, both sides are true, and $T \equiv T$.

We've shown that, no matter which logical value r has, the two sides of our expression are equivalent. **Now** we're done! All it took, in this example, was knowledge of the truth table for implication, and double negation.

Is there a nicer, neater way to write this? Sorry, not really. What I've written above is pretty much what you need to write, because you have to explain to the reader why both sides are equivalent for both values of the letter you chose. You don't have to be quite so expansive with your explanations as I was, but you still need to clearly justify your reasoning.

Might you run into a dead end, as is possible with sequences of logical equivalences? Yes, you certainly might. If that happens, try a different letter. Even doing that might not help. Not every technique is a good choice for solving a problem, just as not every tool in a carpenter's toolbox is a good choice to cut a piece of wood. But, if we ask you to show something by reasoning, you can be sure that we tried it ourselves first, and believe that it can be done.

Looking for more examples of reasoning? Rosen isn't a good place to look, but I do cover the technique in "Kneel Before Zodd."