Proofs of $p \rightarrow q$

Handful O’ Definitions (1 / 2)

**Definition: Conjecture**

**Definition: Theorem**

**Definition: Proof**
Handful O’ Definitions (2 / 2)

Definition: Lemma

Definition: Corollary

Example(s):

Proofs – CSc 245 v1.1 (McCann) – p. 3

Why do People Fear Proofs?

1. Proofs don’t come from an assembly line.
   - Need knowledge, persistence, and creativity

2. Creating proofs seems like magic.
   - But they are systematic in many ways

3. Proofs are hard to read and understand.
   - Only if the writer makes them so

4. Institutionalized Fear.
   - Many teachers avoid them in classes
Constructing a proof? Remember:

1. There are several proof techniques for a reason.
   - One may be easier to use than the others

2. Knowledge of mathematics is important.
   - Remember our Math Review?

3. There are “tricks” to know.
   - Ex: Dividing an even # in half leaves no remainder

4. Practice helps . . . a lot!
   - Just as it does for most everything else

5. Dead ends are expected.
   - Proofs in books are the final, polished versions

Types Of Proof In This Class

1. Direct Proof
   - The most common variety

2. Proof by Contraposition
   - Like Direct, but with a twist

3. Proof by Contradiction
   - A dark road on a foggy night

4. Proof by Mathematical Induction
   - Wait for it . . .
Our First Conjecture

Conjecture: If \( n \) is even, then \( n^2 \) is also even, \( n \in \mathbb{Z} \).
Proof-Writing Miscellanea

- Remember: A conjecture isn’t a theorem until proven.
- Don’t lose sight of your destination.
- When writing proofs in this class:
  1. Always start with “Proof (style):”
  2. Stating your allowed assumptions can help.
  3. Define all introduced variables.
  4. End proofs with “Therefore, ” and the conjecture.

[Outside of this class: “Q.E.D.” (quod erat demonstrandum, Latin for “this was to be demonstrated.”)]

A Conjecture About Inequalities

Conjecture: If \(0 < a < b\), then \(a^2 < b^2\), \(a, b \in \mathbb{R}\).
**Proof By Cases**

**Question:** How would you prove that $\forall x C(x)$ is true, where $x \in \{6, 28, 496\}$?

**A Direct Proof Employing Cases**

**Conjecture:** $s \rightarrow r \equiv \neg r \rightarrow \neg s$.

**Proof (direct):** Consider all possible combinations of values of $r$ and $s$:

<table>
<thead>
<tr>
<th>$r$</th>
<th>$s$</th>
<th>$s \rightarrow r$</th>
<th>$\neg r \rightarrow \neg s$</th>
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</thead>
<tbody>
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Therefore, $s \rightarrow r \equiv \neg r \rightarrow \neg s$.

(Yes, this truth table is a direct proof by cases.)
**Conjecture:** \( x^2 \% 4 \in \{0, 1\}, \ x \in \mathbb{Z}. \)

**Proof or Goof?:**

Consider \( x \) such that \( 0 < x < 1 \). Take the base–10 logarithm of both sides of \( x < 1 \): \( \log_{10} x < \log_{10} 1 \). By definition, \( \log_{10} 1 = 0 \). Divide both sides by \( \log_{10} x \):

\[
\frac{\log_{10} x}{\log_{10} x} < \frac{0}{\log_{10} x},
\]

which reduces to \( 1 < 0 \).

Therefore, \( 1 < 0 \).
**Conjecture:** For all $n \in \mathbb{Z}^{\text{odd}}$, $(n^2 - 1) \% 4 = 0$.

**Proof or Goof?:**
Let $x = 1$. $1^2 - 1 = 0$. $0 \% 4 = 0$. Let $x = 3$. $3^2 - 1 = 8$. $8 \% 4 = 0$. Let $x = 5$. $5^2 - 1 = 24$. $24 \% 4 = 0$. This shows no sign of failing to give a result of 0.

Therefore, for all $n \in \mathbb{Z}^{\text{odd}}$, $(n^2 - 1) \% 4 = 0$.

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**Proof by Contraposition**

*(a.k.a. Proof of the Contrapositive)*
Example #1: Proof by Contraposition

Conjecture: If \( ac \leq bc \), then \( c \leq 0 \), when \( a > b \).

Example #2: Proof by Contraposition

Conjecture: If \( n^2 \) is even, then \( n \) is even.
Proof by Contradiction

(a.k.a. Reductio ad Absurdum)

Recall the Law of Implication: \( p \rightarrow q \equiv \neg p \lor q \)

Example #1: Proof by Contradiction

**Conjecture:** If \( 3n + 2 \) is odd, then \( n \) is odd.
Example #2: Proof by Contradiction (1 / 2)

Conjecture: The sum of the squares of two odd integers is never a perfect square. (Or: If \( n = a^2 + b^2 \), then \( n \) is not a perfect square, where \( a, b \in \mathbb{Z}^{\text{odd}} \).)

Example #2: Proof by Contradiction (2 / 2)
How To Prove Biconditional Expressions

(i.e., Conjectures Of The Form $p \leftrightarrow q$)

Example(s):

Disproving Conjectures

Typical approaches:

(1) Prove that the conjecture’s negation is true.

(2) Find a counter-example. (Very commonly used!)

Example(s):