Structural Induction

Slides are by courtesy of Russell Lewis.
Many of the discrete structures we use can be recursively characterized or recursively defined.

That is good, because a recursive definition lets us prove important properties about these structures using induction.
Structural Induction  
(over non-empty lists)

If we can prove some property for a list of length 1, 

. . . and also prove it for a longer list, assuming it's true for a slightly shorter one, 

. . . then it is true for all nonempty lists.
Structural Induction  
(over non-empty lists)

- A non-empty linked list can be defined as (basis) a single node, or (recursive step) a node plus a link to a nonempty linked list.
- **Theorem**: a non-empty linked list with \( k \) nodes has \( k-1 \) links.
- **Proof** (by structural induction):
  - **Basis**: a LL with a single node (\( k=1 \)) has no links, that is, \( 0 = k-1 \) links.
Structural Induction
(over non-empty lists)

- Inductive step: assume the claim holds for any nonempty linked list of \( m \) nodes.
- We wish to show that the claim holds for a list \( L \) consisting of \( m+1 \) nodes, i.e., that list \( L \) has \((m+1)-1\) links, or \( m \) links.
- Since the size of \( L \) exceeds 1, our definition of nonempty linked lists says that \( L \) must be a single node linked to a nonempty linked list (call it \( J \)).
- Since \( J \) has length \( m \), it must have \( m-1 \) links, by the I.H. So \( L \) has \( m-1+1 = m \) links.
Structural Induction
(over non-empty lists)

Recursive definition (or characterization) and structural induction work hand in hand.

If structural induction shows the desired property for all base-cases and all recursive steps of the definition, then the property holds for all the structures covered by the definition.
Structural Induction over trees

Trees are recursively defined: for each node, its subtrees are also trees.
Structural Induction over trees: basis

If we can prove the claim for a tree of one node (height of zero),

and, if necessary, for an empty tree,
Structural Induction over trees: inductive step

... and also prove the claim for an arbitrary nonempty tree (assuming the claim holds for the root's subtrees) ...

then by induction, the claim holds for any tree.
Structural Induction Over Trees

- Your basis might cover an empty tree, or a single node (for claims about nonempty trees).
- The inductive step might be one of three common strategies:
  - root + subtrees
  - add one more leaf
  - (complete trees) add a whole row of leaves
- The first is the most common, forget the rest.
Claim: In a non-empty, complete binary tree of height $h$, the number of leaves is $2^h$.

We will define non-empty complete binary trees using the root+subtrees definition.

Proof: in class
More Structural Induction Practice

- Prove that in any nonempty complete binary tree, the number of internal nodes is $2^h - 1$.
- Prove that in any nonempty tree (not necessarily binary), that the number of nodes exceeds the number of links by one.