Homework 2 Solutions

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Problem 1

```java
min = max = list[0];
o+=4
for(int i = 1; i < n-1; i+=2){
o+=4
    if(list[i] < list[i+1]){  
        if(list[i] < min) o+=3
            min = list[i]; o+=3
        if(list[i+1] > max) o+=4
            max = list[i+1]; o+=4
    } else {
        if(list[i+1] < min) o+=4
            min = list[i+1]; o+=4
        if(list[i] > max) o+=3
            max = list[i]; o+=3
    }
o+=2
}
o+=2
if(i == n-1){
o+=2
    if(list[i] < min) o+=3
        min = list[i]; o+=3
    if(list[i] > max) o+=3
        max = list[i]; o+=3
}

Total: 24(n-2/2) + +21
= 12n - 24 + 21
= 12n - 3
```

Problem 2

\[ 2 < \log_3 n < \log_2 n < n^{\frac{2}{3}} < 20n < 4n^2 < 3^n < n! \]  \hspace{1cm} (1)

Problem 3

To prove that \( f(n) \in O(n) \), show \( f(n) \leq c(g(n)) \forall n_0 \geq n \)

To prove that \( f(n) \in \Omega(n) \), show \( f(n) \geq c(g(n)) \forall n_0 \geq n \)

1. \( c_1 n \)
   (a) \( O(n) \): Choose a \( c \) such that \( c > c_1 \)
   (b) \( \Omega(n) \): Choose a \( c \) such that \( c < c_1 \)

2. \( c_2 n^3 + c_3 \)
(a) \(O(n)\): we know that by the polynomial theorem that \(c_2 n^3 + c_3 = O(n^3)\)

(b) \(\Omega(n)\): We want \(c_2 n^3 + c_3 \geq cn^3\), so solve \(\frac{c_2 n^3 + c_3}{n^3} \geq c\). Choosing \(n_0\) to be any number, say 1, gives \(c_2 \geq c\) (this is true because \(\frac{c_2}{n^3}\) becomes smaller as \(n\) increases). We apply similar logic for the rest of the problems.

3. \(c_4 n \log n + c_5 n\)

   (a) \(O(n)\): \(c_4 n \log n + c_5 n \leq cn\log n\), \(\frac{c_4 n \log n + c_5 n}{n \log n} \leq c\). Choose \(n_0 = 2\) and \(c_4 + c_5 \leq c\)

   (b) \(\Omega(n)\): \(c_4 n \log n + c_5 n \geq cn\log n\), \(\frac{c_4 n \log n + c_5 n}{n \log n} \geq c\). Choose \(n_0 = 2\) and \(c_4 \geq c\)

4. \(c_6 2^n + c_7 n^6\)

   (a) \(O(n)\): \(c_6 2^n + c_7 n^6 \leq c 2^n\), \(\frac{c_6 2^n + c_7 n^6}{2^n} \leq c\). Using l’Hôpital’s rule, we find that \(\lim_{n \to \infty} \frac{c_6 n^6}{2^n} = 0\), which says that \(2^n\) grows faster than \(n^6\). So, just find an \(n_0\) such that \(n^6 \leq 2^n\). \(n_0 = 30\).

   (b) \(\Omega(n)\): \(c_6 2^n + c_7 n^6 \geq c 2^n\), \(c_6 + \frac{c_7 n^6}{2^n} \geq c\). Pick \(c \leq c_6 + c_7\). We know from our Big-O calculations that \(\frac{c_7 n^6}{2^n} < 1\) after \(n_0 = 30\).

**Problem 4**

1. (c) \(O(n^2)\)

2. (c) \(O(n \log n)\)

3. (g) \(O(n^3 \log n)\)

4. (i) \(O(n)\)

**Problem 5**

We can use the Master Theorem because \(a \geq 1, b > 1, c > 0, \) and \(k \geq 0\). Using the Master Theorem: \(a = 1, b = 2, \) \(c = 1, k = \frac{1}{2}\), so \(1 < \sqrt{2}\), so this is \(O(\sqrt{n})\)

**Problem 6**

\[
T(n) = 2T\left(\frac{n}{2}\right) + n, T(2) = 2
\]

\[
T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{2^2}\right) + \frac{n}{2}
\]

\[
T\left(\frac{n}{2^2}\right) = 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}
\]

\[
T\left(\frac{n}{2^3}\right) = 2(2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}) + \frac{n}{2}
\]

\[
T\left(\frac{n}{2^3}\right) = 2^2T\left(\frac{n}{2^3}\right) + 2\frac{n}{2^2} + \frac{n}{2}
\]

\[
T(n) = 2^2T\left(\frac{n}{2^3}\right) + 2\frac{n}{2^2} + \frac{n}{2} + n
\]

\[
T(n) = 2^3T\left(\frac{n}{2^3}\right) + 3n
\]

\[
T(n) = 2^kT\left(\frac{n}{2^k}\right) + kn
\]

Setting \(\frac{n}{2^k} = 2\), \(n = 2^{k+1}\), \(\log_2 n = k + 1\), \(\log_2 n - 1 = k\)

\[
T(n) = 2^{k+1} - 1 + (\log_2 n - 1) n
\]

\[
T(n) = n + n\log_2 n - n
\]
$T(n) = n \log_2 n$

According to Master Theorem, $T(n) = O(n \log n)$

Proof (Inductive):
Basis: $n=2$
$T(2) = 2$ by definition. And our closed form says $n \log n = 2 \times \log_2 2 = 2$

Inductive: $T(n) = n \log n \rightarrow T(2n) = (2n \log(2n))$

$T(2n) = 2T(n) + n$

From IH, $T(2n) = 2n \log n + 2n$
$T(2n) = 2n(\log n + 1)$
$T(2n) = 2n \log 2n$

QED