Transformations

\[
\begin{bmatrix}
\cos \theta & \sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}
= \begin{bmatrix}
a_1' \\
a_2'
\end{bmatrix}
\]
Transformations

A transformation \( \tau \) is a mapping of \( \mathbb{R}^n \) to itself (not necessarily one-to-one. Many points might be transformed to the same point.)

Affine transformation - \( \tau(V) = AV + b \)
- \( A \) - matrix
- \( b \) - scalar

Meaningful only for vectors. For point \( P \), \( \tau(P) \) is a transformation of the vector from the origin to \( P \).

Scaling

- \( V = (v_x, v_y) \) - vector in XY plane

Scaling operator \( S \) with parameters \((s_x, s_y)\):

\[
S^{(s_x, s_y)}(V) = \left( v_x s_x, v_y s_y \right)
\]

Scaling

- Matrix form:

\[
S_{(s_x, s_y)}(V) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = (v_x s_x, v_y s_y)
\]

- Independent in \( x \) and \( y \)
- Non-uniform scaling is allowed
- What is the meaning of scaling by zero?

Polar form of a point

- Polar form:

\[
V = (v_x, v_y) = (r \cos \alpha, r \sin \alpha)
\]

\[
V_2 = r \cos \alpha
\]

\[
V_2 = r \sin \alpha
\]

- \( r \) is the distance of \( V \) from the origin (0,0)

\[
r = \sqrt{v_x^2 + v_y^2}
\]

Rotation

- Polar form:

\[
V = (v_x, v_y) = (r \cos \alpha, r \sin \alpha)
\]

- Rotate \( V \) anti-clockwise by \( \theta \) to \( W \):
Rotation

- Polar form:
  \[ V = (v_x, v_y) = (r \cos \alpha, r \sin \alpha) \]
- Rotate \( V \) counter-clockwise by \( \theta \) to \( W \):
  \[ W = (w_x, w_y) = (r \cos(\alpha + \theta), r \sin(\alpha + \theta)) = (r \cos \alpha \cos \theta - r \sin \alpha \sin \theta, r \cos \alpha \sin \theta + r \sin \alpha \cos \theta) \]

Rotation Properties

- \( R^\theta \) is orthogonal
  \[ [R^\theta]^T = [R^\theta] \]
- Rotation by \( -\theta \) is
  \[ R^{-\theta}(V) = (v_x, v_y) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} [R^\theta]^T \]

Translation

- Translation operator \( T \) with parameters \((t_x, t_y)\):
  \[ T(t_x, t_y) (V) = (v_x + t_x, v_y + t_y) \]
- Can we express \( T \) in a matrix form?

Translation - Homogeneous Coordinates

- To represent \( T \) in matrix form - use homogeneous coordinates:
  \[ V^h = [v_x, v_y, 1] \rightarrow [v_x', v_y', 1] \]
- Conversion (projection) from homogeneous to Euclidean:
  \[ V = (v_x, v_y) = \begin{bmatrix} v_x' \\ v_y' \\ 1 \end{bmatrix} \]
- In homogeneous coordinates:
  \[ (2, 2, 1) \rightarrow (4, 4, 2) = (1, 1, 0.5) \]

Translation

- Using homogeneous coordinates:
  \[ T(t_x, t_y)(V^h) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = (v_x + t_x, v_y + t_y, 1) \]
Why is it useful to use Matrix form to represent the transformations?

Transformation Composition

What operation rotates \( \theta \) around \( P = (p_x, p_y) \)?
- Translate \( P \) to origin
- Rotate around origin by \( \theta \)
- Translate back

Transformation Composition

\[
T^{T', P', T'} \cdot R \cdot T^{P, P} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
p_x & p_y & 1
\end{bmatrix}
\]

Let \( Q \) denote this matrix (computed once).

For every point \( p \) of the many points of the “house”, we apply \( p' = Qp \)
(read: the old corner \( p \) is transformed to the new corner \( p' \))

Transformations Quiz

- What do these transformations do?
- And these homogeneous ones?
- How can one reflect a planar object through an arbitrary line in the plane?
- Can one rotate a planar object in the plane by reflection?

Arbitrary Reflection

Shift by \((0, b)\)
Rotate by \(\alpha\)
Reflect through \(x\)
Rotate by \(\alpha\)
Shift by \((0, b)\)

Rotation Approximation

For small angles can approximate:
- \(x' = x \cos \theta - y \sin \theta\)
- \(y' = x \sin \theta + y \cos \theta\)

Since \(\cos \theta \to 1\) and \(\sin \theta \to \theta\) as \(\theta \to 0\) (Taylor expansion of \(\sin\theta\))

Example (steps of \(\theta\)):

- When is this rotation approximation useful?