Exercies #2 in Computer Graphics Algorithms and Mathematics

Foley 3.1 It is a quite trivial task to implement the algorithm in the case that the slope of the line is 1. This is just the loop *for(*$x = x_0, y = y_0$ *;* $x \leq x_1$*;* $x++, y++$*) putpixel(*x, y*) ;*

Foley 3.3 A formal proof can be given by induction. We assume that the line ℓ passes above the previous pixel (South-West). Since the slope is the line (by our assumption) is between 0 and 1, it must pass between the 'E' and the 'NE' corners. Since the distance between these points is 1, and we set the next pixel to be the one closer to the midpoint, the distance must be $\leq 1/2$.

Foley 5.1 Let M be any matrix, let $\mathbf{p}_1, \mathbf{p}_2$ be points in 3D. The segment connecting $\mathbf{p}_1$ and $\mathbf{p}_2$ can be represented as $\{t\mathbf{p}_1 + (1-t)\mathbf{p}_2 \mid 0 \leq t \leq 1\}$. We need to show that every point on the segment connecting $\mathbf{p}_1$ and $\mathbf{p}_2$ is transformed into a point on the segment connecting $M\mathbf{p}_1$ and $M\mathbf{p}_2$, and vice versa. This is obvious, since

$$M(t\mathbf{p}_1 + (1-t)\mathbf{p}_2) = tM\mathbf{p}_1 + (1-t)M\mathbf{p}_2 \ ,$$

due to linearity of matrix multiplication.

Foley 5.3 Let $\vec{v} = (a, b, 1)^T$. Let

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad S(\alpha, \beta) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$$

Then

$$\begin{aligned}\vec{v}_1 = \; & R(\theta) \cdot S(\alpha,\beta)\vec{v} \; = & (\alpha a \cos\theta - \beta b \sin\theta,\; \alpha a \sin\theta + \beta b \cos\theta) \\ \vec{v}_2 = \; & S(\alpha,\beta) \cdot R(\theta)\vec{v} \; = & (\alpha(a\cos\theta - \; b\sin\theta),\; \beta(a\sin\theta + b\cos\theta)\;)\end{aligned}$$

Thus $\vec{v}_1 - \vec{v}_2 = ((\alpha - \beta)b\sin\theta, (\alpha-\beta)a\sin\theta)$. Clearly if $\alpha = \beta$ then $\vec{v}_1 - \vec{v}_2 = (0,0)$, that is $\vec{v}_1 = \vec{v}_2$. Similarly, if $\theta = n\pi$, then $\sin\theta = 0$, and again $\vec{v}_1 = \vec{v}_2$.

On the other hand, if $\alpha \neq \beta$ and $\sin\theta \neq 0$, then at least one of the components of $\vec{v}_1 - \vec{v}_2$ are not zero, as required.

Foley 5.7 Let d be the diagonal of the cube (connecting $(0,0,0)$ to $(1,1,1)$). Let d_p be the projection of d on the xz-plane. Clearly $d_p = (1,0,1)$. The distance of d from $(0,0,0)$ is $\sqrt{3}$ and the distance of d_p from $(0,0,0)$ is $\sqrt{2}$. We first rotate the cube along the y axis by 45 degrees $R_y(45)$. Let d' and d'_p denote d and d_p after the first rotation. After the transformation, both d' and d'_p are on the $yz = 0$ plane, and d'_p is on the z-axis; $d' = (0,1,\sqrt{2})$, and $d'_p = (0,0,\sqrt{2})$. We next rotate the cube around the x-axis by α degrees, so that d' would be on the z-axis. To compute α, note that α is one angle of triangle forms by $d', (0,0,0), d'_d$, so $\cos\alpha = \frac{\sqrt{2}}{\sqrt{3}}$ and $\sin\alpha = \frac{1/}{\sqrt{2}}$. Thus determines $R_x(\alpha)$.

Once d' is on the x-axis, we rotate around the z-axis by θ degrees, and apply the inverse transformations to return d to $(1,1,1)$. So the overall transformation is $R = R_y(-45) \cdot R_x(-\alpha) \cdot R_z(\theta) \cdot R_x(\alpha) \cdot R_y(45)$.

Foley 5.11 We first rotate by α degrees around the z axis, so U is shifted into a point U' on the yz plane. To find α, look at the triangle $\triangle U(0,0,u_z)(0,u_y,u_z)$. This triangle lies on the plane $z = u_z$. Every rotation around the z-axis would send U into a point on this plane. Let $\alpha = \angle U(0,0,u_z)(0,u_y,u_z)$. Let $d = \sqrt{u_x^2 + u_y^2}$. Clearly $\cos\alpha = u_y/d$ and $\sin\alpha = u_x/d$. This roration brings U into the point $U' = (0,d,u_z)$ on the yz-plane.

Next we need to rotate around the x-axis by β degrees, so U' would be rotated into a point U'' on the z-axis. Consider $D = |U| = |U'| = \sqrt{u_x^2 + u_y^2 + y_z^2}$. Finding β is easy. Think of the triangle $\triangle U'(0,0,0)(0,0,u_z)$. In this triangle, consider the angle $\beta = \angle U', (0,0,0), (0,0,u_z)$. Thus $\cos\beta = u_z/D$ and $\sin\beta = d/D$.

Foley 6.3 A point $p = (x_0, y_0, z_0)$ of a polygon will be clipped if and only if it is outside the faces parallel to the z-axis of the clipping volume. Since projecting p on the xy plane would preserve its (x, y) coordinate, this implies that the order is insignificant.