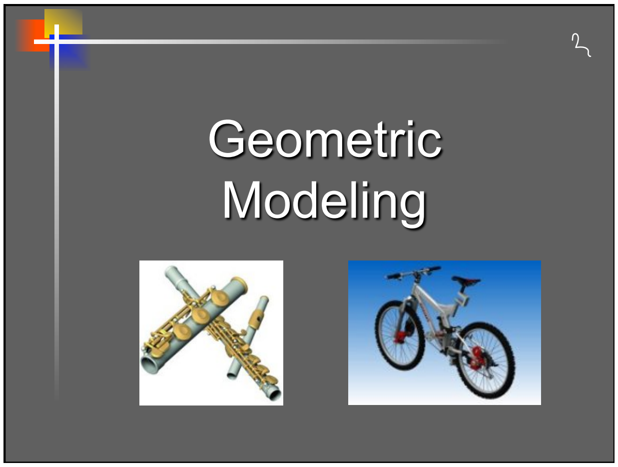
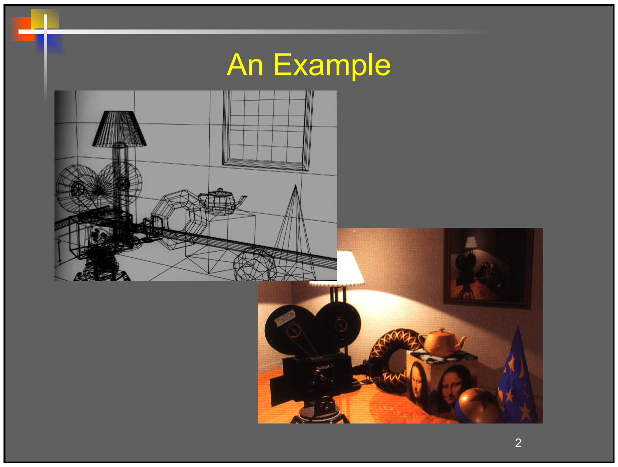
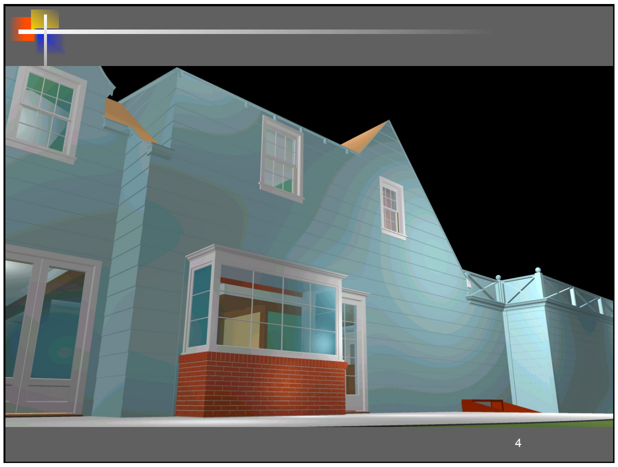
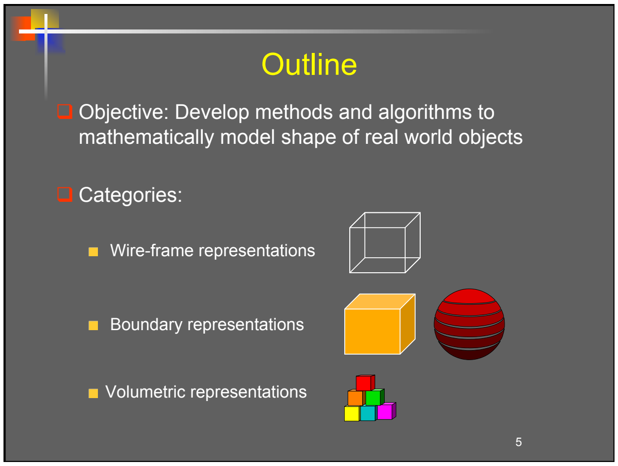


Geometric Modeling



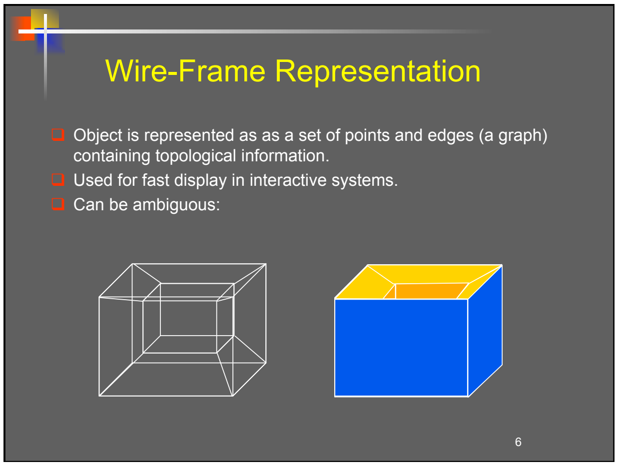
An Example





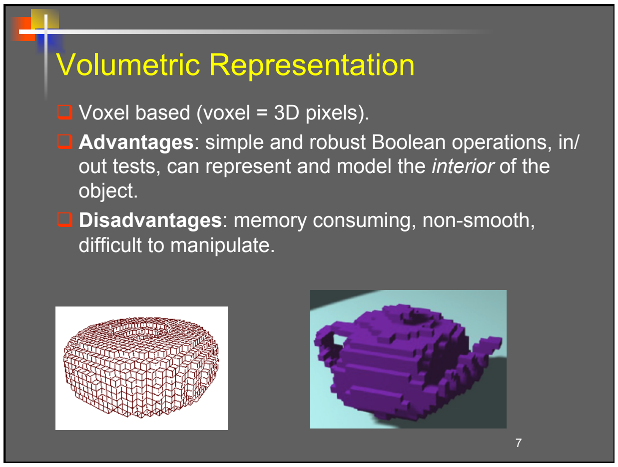
Outline

- Objective: Develop methods and algorithms to mathematically model shape of real world objects
- Categories:
  - Wire-frame representations
  - Boundary representations
  - Volumetric representations



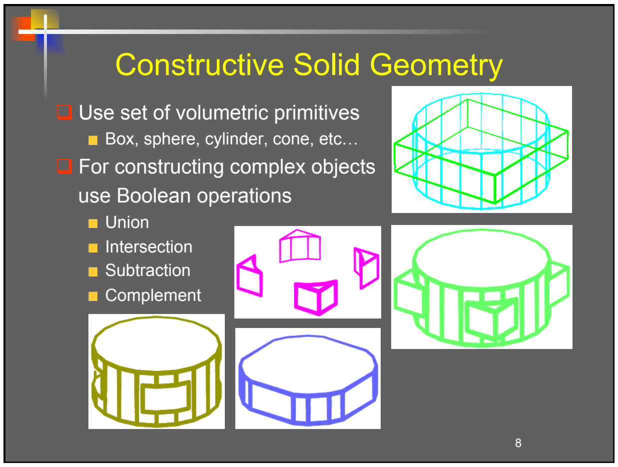
Wire-Frame Representation

- Object is represented as as a set of points and edges (a graph) containing topological information.
- Used for fast display in interactive systems.
- Can be ambiguous:



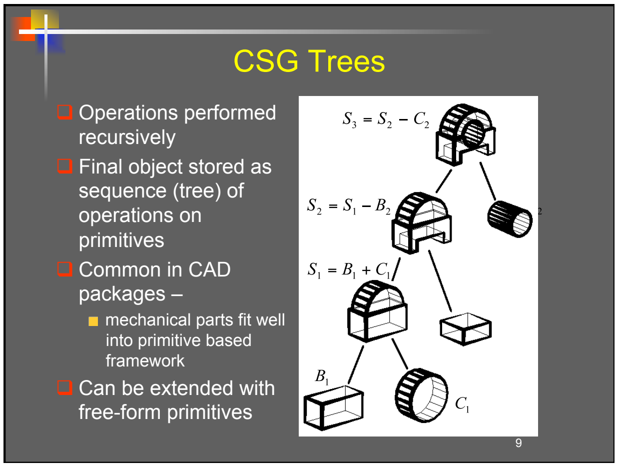
Volumetric Representation

- Voxel based (voxel = 3D pixels).
- **Advantages**: simple and robust Boolean operations, in/out tests, can represent and model the *interior* of the object.
- **Disadvantages**: memory consuming, non-smooth, difficult to manipulate.



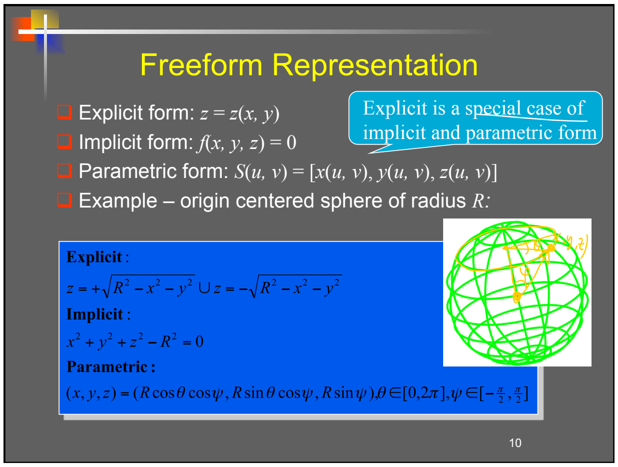
Constructive Solid Geometry

- Use set of volumetric primitives
  - Box, sphere, cylinder, cone, etc…
- For constructing complex objects use Boolean operations
  - Union
  - Intersection
  - Subtraction
  - Complement



CSG Trees

- Operations performed recursively
- Final object stored as sequence (tree) of operations on primitives
- Common in CAD packages –
  - mechanical parts fit well into primitive based framework
- Can be extended with free-form primitives



Freeform Representation

- Explicit form: $z = z(x, y)$
- Implicit form: $f(x, y, z) = 0$
- Parametric form: $S(u, v) = [x(u, v), y(u, v), z(u, v)]$
- Example – origin centered sphere of radius R:

Explicit is a special case of implicit and parametric form

Explicit :

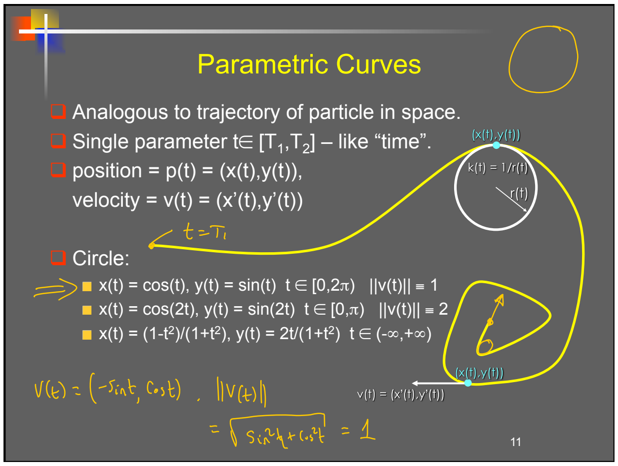
$$z = +\sqrt{R^2 - x^2 - y^2} \cup z = -\sqrt{R^2 - x^2 - y^2}$$

Implicit :

$$x^2 + y^2 + z^2 - R^2 = 0$$

Parametric :

$$(x, y, z) = (R\cos\theta\cos\psi, R\sin\theta\cos\psi, R\sin\psi), \theta \in [0, 2\pi], \psi \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$



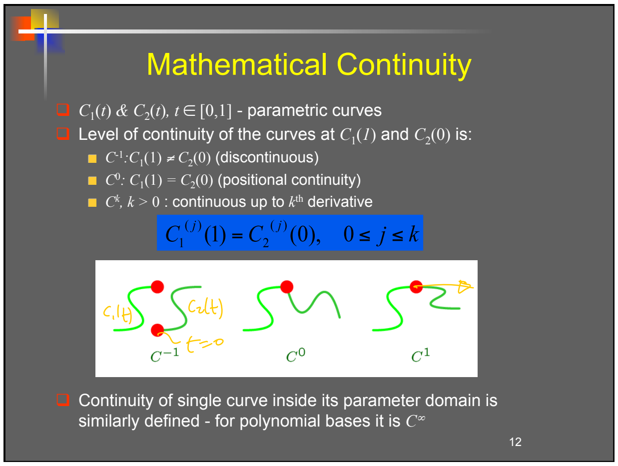
Parametric Curves

- Analogous to trajectory of particle in space.
- Single parameter $t \in [T_1, T_2]$ – like "time".
- position = $p(t) = (x(t), y(t))$,
  velocity = $v(t) = (x'(t), y'(t))$
- Circle:
  - $x(t) = \cos(t)$, $y(t) = \sin(t)$ $t \in [0, 2\pi)$ $\|v(t)\| \equiv 1$
  - $x(t) = \cos(2t)$, $y(t) = \sin(2t)$ $t \in [0, \pi)$ $\|v(t)\| \equiv 2$
  - $x(t) = (1-t^2)/(1+t^2)$, $y(t) = 2t/(1+t^2)$ $t \in (-\infty, +\infty)$

$t = T_1$

$V(t) = (-\sin t, \cos t)$. $\|V(t)\|$

$= \sqrt{\sin^2 t + \cos^2 t} = 1$

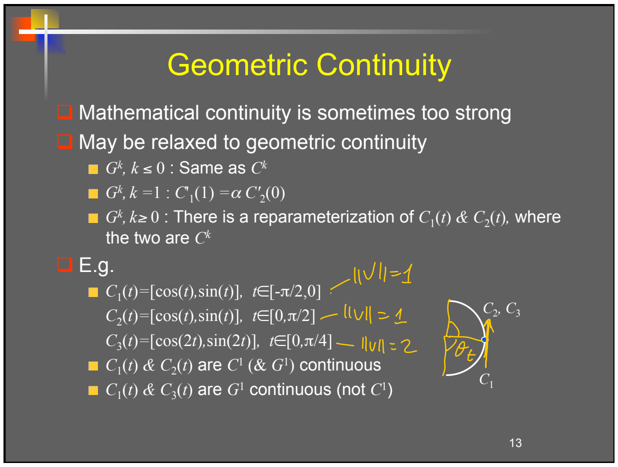


Mathematical Continuity

- $C_1(t)$ & $C_2(t)$, $t \in [0,1]$ - parametric curves
- Level of continuity of the curves at $C_1(1)$ and $C_2(0)$ is:
  - C^{-1}: $C_1(1) \neq C_2(0)$ (discontinuous)
  - C^0: $C_1(1) = C_2(0)$ (positional continuity)
  - C^k, $k > 0$: continuous up to k^{th} derivative

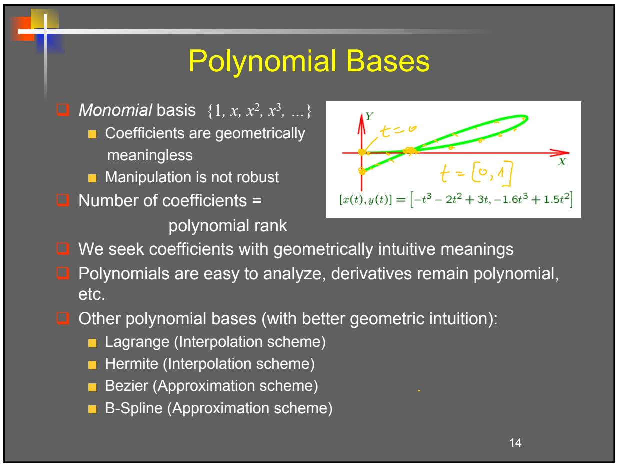
$$C_1^{(j)}(1) = C_2^{(j)}(0), \quad 0 \leq j \leq k$$

- Continuity of single curve inside its parameter domain is similarly defined - for polynomial bases it is C^∞



Geometric Continuity

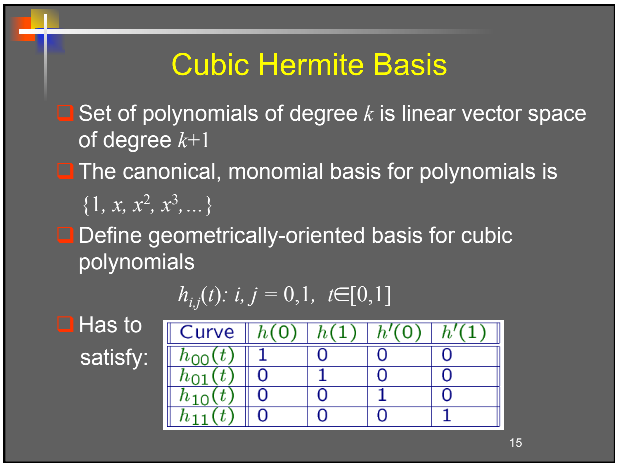
- Mathematical continuity is sometimes too strong
- May be relaxed to geometric continuity
  - $G^k, k \le 0$: Same as C^k
  - $G^k, k = 1 : C'_1(1) = \alpha\, C'_2(0)$
  - $G^k, k \ge 0$: There is a reparameterization of $C_1(t)$ & $C_2(t)$, where the two are C^k
- E.g.
  - $C_1(t)=[\cos(t),\sin(t)],\ t\in[-\pi/2,0]$ $\|V\|=1$
    $C_2(t)=[\cos(t),\sin(t)],\ t\in[0,\pi/2]$ — $\|V\|=1$
    $C_3(t)=[\cos(2t),\sin(2t)],\ t\in[0,\pi/4]$ — $\|V\|=2$
  - $C_1(t)$ & $C_2(t)$ are C^1 (& G^1) continuous
  - $C_1(t)$ & $C_3(t)$ are G^1 continuous (not C^1)



Polynomial Bases

- *Monomial* basis $\{1, x, x^2, x^3, \ldots\}$
  - Coefficients are geometrically meaningless
  - Manipulation is not robust
- Number of coefficients = polynomial rank
- We seek coefficients with geometrically intuitive meanings
- Polynomials are easy to analyze, derivatives remain polynomial, etc.
- Other polynomial bases (with better geometric intuition):
  - Lagrange (Interpolation scheme)
  - Hermite (Interpolation scheme)
  - Bezier (Approximation scheme)
  - B-Spline (Approximation scheme)

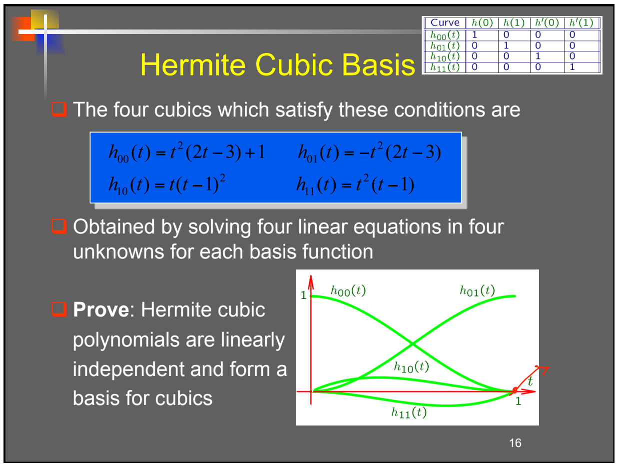
$[x(t),y(t)] = \left[-t^3-2t^2+3t,\ -1.6t^3+1.5t^2\right]$



Cubic Hermite Basis

- Set of polynomials of degree k is linear vector space of degree k+1
- The canonical, monomial basis for polynomials is
  $\{1, x, x^2, x^3, \ldots\}$
- Define geometrically-oriented basis for cubic polynomials
  $h_{i,j}(t)$: $i, j = 0,1,\ t\in[0,1]$
- Has to satisfy:

Curve	$h(0)$	$h(1)$	$h'(0)$	$h'(1)$
$h_{00}(t)$	1	0	0	0
$h_{01}(t)$	0	1	0	0
$h_{10}(t)$	0	0	1	0
$h_{11}(t)$	0	0	0	1



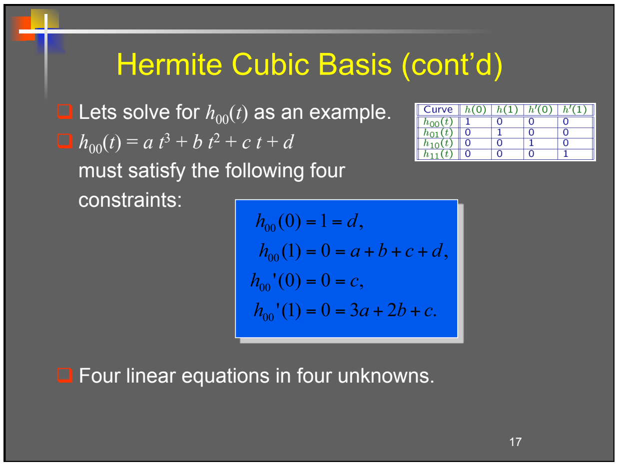
Hermite Cubic Basis

Curve	$h(0)$	$h(1)$	$h'(0)$	$h'(1)$
$h_{00}(t)$	1	0	0	0
$h_{01}(t)$	0	1	0	0
$h_{10}(t)$	0	0	1	0
$h_{11}(t)$	0	0	0	1

- The four cubics which satisfy these conditions are

$$h_{00}(t)=t^2(2t-3)+1 \qquad h_{01}(t)=-t^2(2t-3)$$
$$h_{10}(t)=t(t-1)^2 \qquad h_{11}(t)=t^2(t-1)$$

- Obtained by solving four linear equations in four unknowns for each basis function
- **Prove**: Hermite cubic polynomials are linearly independent and form a basis for cubics



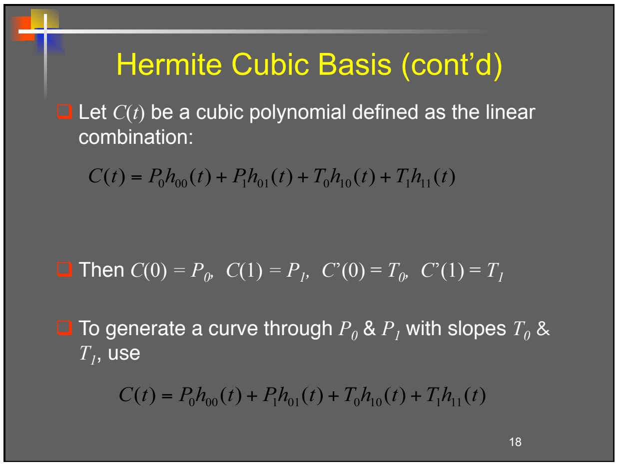
Hermite Cubic Basis (cont'd)

Curve	$h(0)$	$h(1)$	$h'(0)$	$h'(1)$
$h_{00}(t)$	1	0	0	0
$h_{01}(t)$	0	1	0	0
$h_{10}(t)$	0	0	1	0
$h_{11}(t)$	0	0	0	1

- Lets solve for $h_{00}(t)$ as an example.
- $h_{00}(t) = a\,t^3 + b\,t^2 + c\,t + d$ must satisfy the following four constraints:

$$\begin{aligned} h_{00}(0) &= 1 = d, \\ h_{00}(1) &= 0 = a+b+c+d, \\ h_{00}'(0) &= 0 = c, \\ h_{00}'(1) &= 0 = 3a+2b+c. \end{aligned}$$

- Four linear equations in four unknowns.



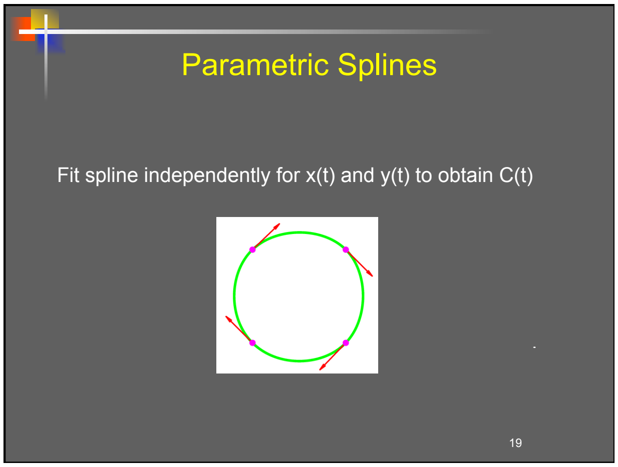
Hermite Cubic Basis (cont'd)

- Let $C(t)$ be a cubic polynomial defined as the linear combination:

$$C(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$

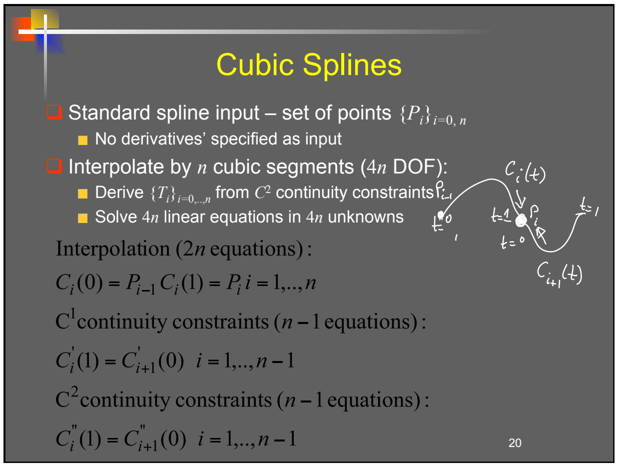
- Then $C(0) = P_0,\ C(1) = P_1,\ C'(0) = T_0,\ C'(1) = T_1$
- To generate a curve through P_0 & P_1 with slopes T_0 & T_1, use

$$C(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$



Parametric Splines

Fit spline independently for x(t) and y(t) to obtain C(t)



Cubic Splines

- Standard spline input – set of points $\{P_i\}_{i=0,n}$
  - No derivatives' specified as input
- Interpolate by n cubic segments ($4n$ DOF):
  - Derive $\{T_i\}_{i=0,\dots,n}$ from C^2 continuity constraints
  - Solve $4n$ linear equations in $4n$ unknowns

Interpolation ($2n$ equations):

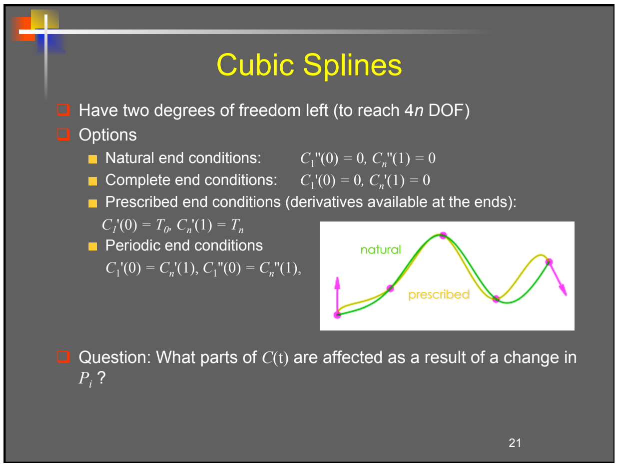
$$C_i(0) = P_{i-1}\; C_i(1) = P_i\; i = 1,..,n$$

C^1 continuity constraints ($n-1$ equations):

$$C_i'(1) = C_{i+1}'(0) \quad i = 1,..,n-1$$

C^2 continuity constraints ($n-1$ equations):

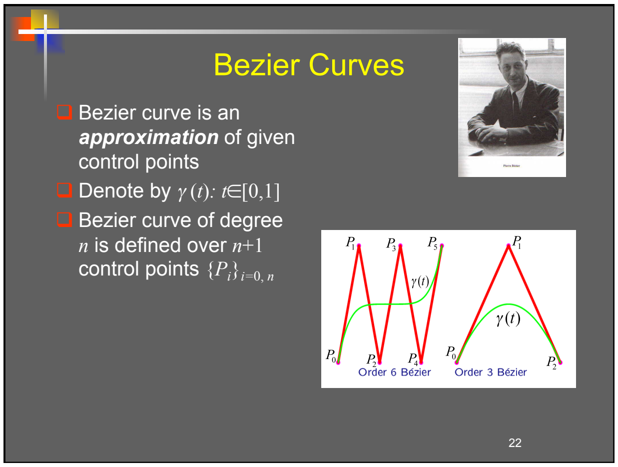
$$C_i''(1) = C_{i+1}''(0) \quad i = 1,..,n-1$$



Cubic Splines

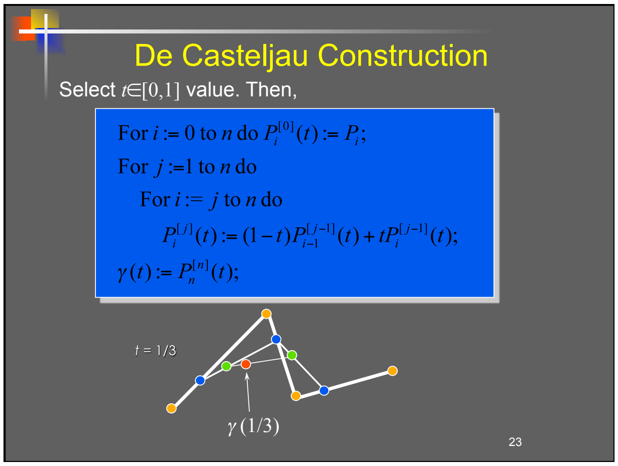
- Have two degrees of freedom left (to reach 4n DOF)
- Options
  - Natural end conditions: $C_1''(0) = 0,\ C_n''(1) = 0$
  - Complete end conditions: $C_1'(0) = 0,\ C_n'(1) = 0$
  - Prescribed end conditions (derivatives available at the ends):
    $C_1'(0) = T_0,\ C_n'(1) = T_n$
  - Periodic end conditions
    $C_1'(0) = C_n'(1),\ C_1''(0) = C_n''(1),$

- Question: What parts of C(t) are affected as a result of a change in P_i ?



Bezier Curves

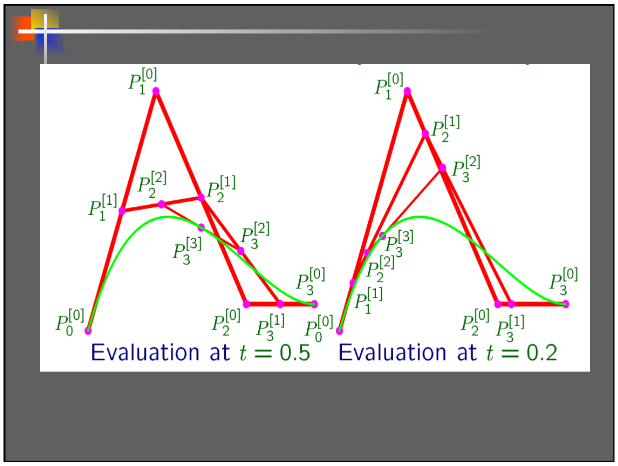
- Bezier curve is an ***approximation*** of given control points
- Denote by $\gamma(t)$: $t\in[0,1]$
- Bezier curve of degree n is defined over n+1 control points $\{P_i\}_{i=0,n}$

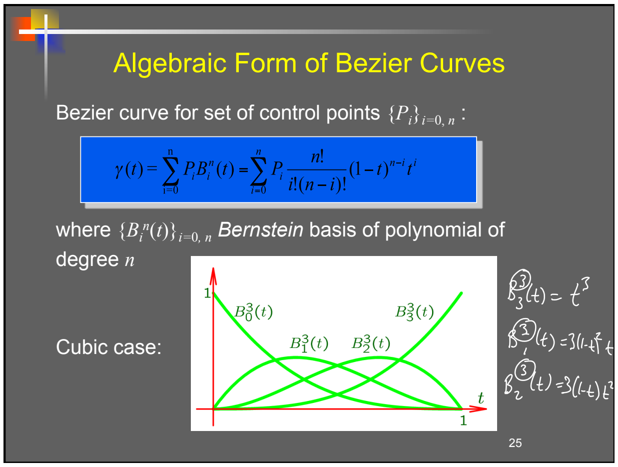


De Casteljau Construction

Select $t\in[0,1]$ value. Then,

```
For $i$ := 0 to $n$ do $P_i^{[0]}(t) := P_i$;
For $j$ :=1 to $n$ do
    For $i$ := $j$ to $n$ do
        $P_i^{[j]}(t) := (1-t)P_{i-1}^{[j-1]}(t) + tP_i^{[j-1]}(t)$;
$\gamma(t) := P_n^{[n]}(t)$;
```





Algebraic Form of Bezier Curves

Bezier curve for set of control points $\{P_i\}_{i=0,n}$:

$$\gamma(t)=\sum_{i=0}^{n} P_i B_i^n(t)=\sum_{i=0}^{n} P_i \frac{n!}{i!(n-i)!}(1-t)^{n-i}t^i$$

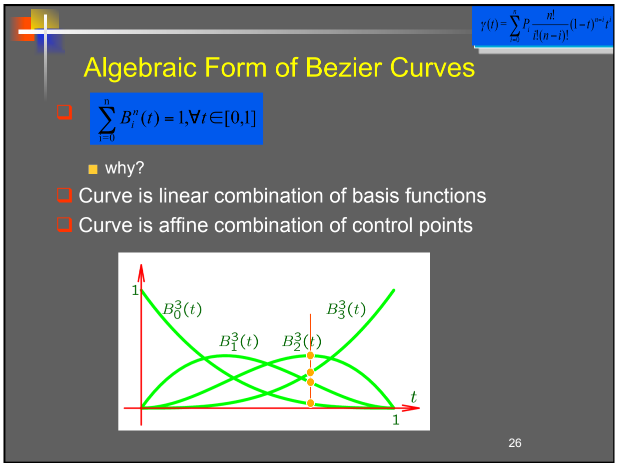
where $\{B_i^n(t)\}_{i=0,n}$ *Bernstein* basis of polynomial of degree n

Cubic case:

$B^{\textcircled{3}}_3(t) = t^3$

$B^{\textcircled{3}}_1(t) = 3(1-t)^2 t$

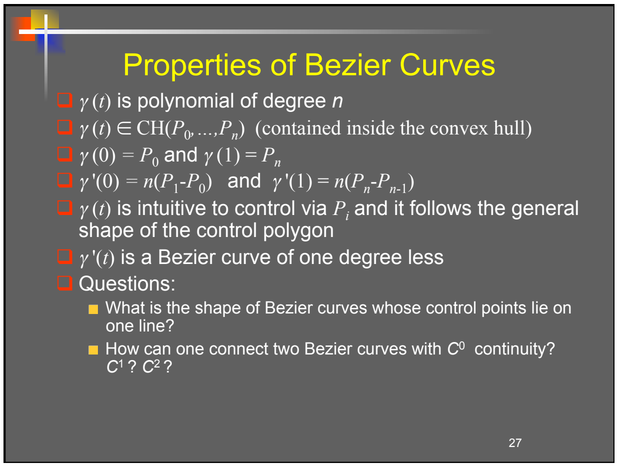
$B^{\textcircled{3}}_2(t) = 3(1-t)t^2$



$\gamma(t)=\sum_{i=0}^{n} P_i \frac{n!}{i!(n-i)!}(1-t)^{n-i}t^i$

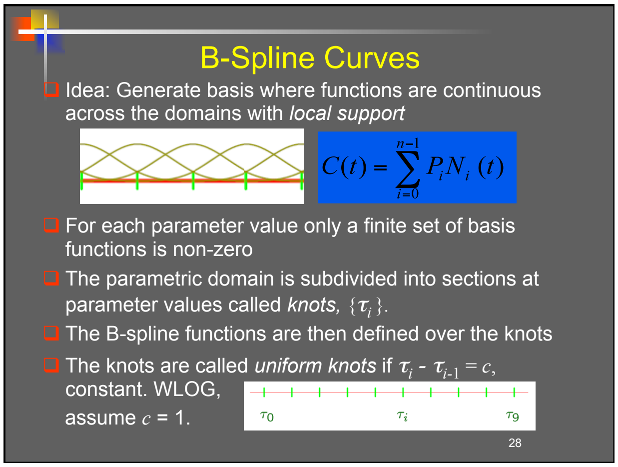
Algebraic Form of Bezier Curves

- $\sum_{i=0}^{n} B_i^n(t) = 1, \forall t \in [0,1]$
  - why?
- Curve is linear combination of basis functions
- Curve is affine combination of control points



Properties of Bezier Curves

- $\gamma(t)$ is polynomial of degree *n*
- $\gamma(t) \in \mathrm{CH}(P_0,\ldots,P_n)$ (contained inside the convex hull)
- $\gamma(0) = P_0$ and $\gamma(1) = P_n$
- $\gamma'(0) = n(P_1\text{-}P_0)$ and $\gamma'(1) = n(P_n\text{-}P_{n-1})$
- $\gamma(t)$ is intuitive to control via P_i and it follows the general shape of the control polygon
- $\gamma'(t)$ is a Bezier curve of one degree less
- Questions:
  - What is the shape of Bezier curves whose control points lie on one line?
  - How can one connect two Bezier curves with C^0 continuity? C^1? C^2?

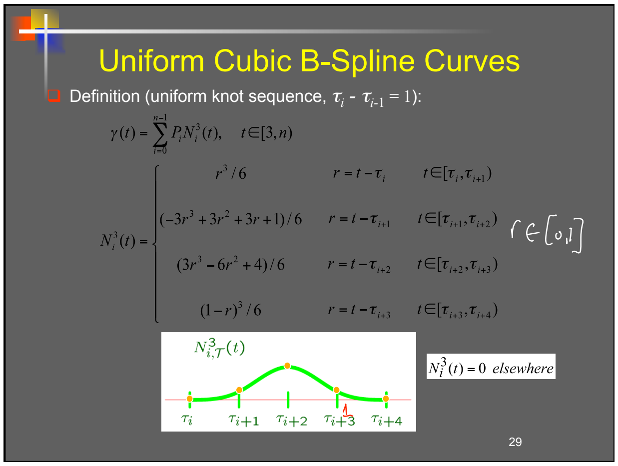


B-Spline Curves

- Idea: Generate basis where functions are continuous across the domains with *local support*

$$C(t) = \sum_{i=0}^{n-1} P_i N_i(t)$$

- For each parameter value only a finite set of basis functions is non-zero
- The parametric domain is subdivided into sections at parameter values called *knots*, $\{\tau_i\}$.
- The B-spline functions are then defined over the knots
- The knots are called *uniform knots* if $\tau_i - \tau_{i-1} = c$, constant. WLOG, assume $c = 1$.



Uniform Cubic B-Spline Curves

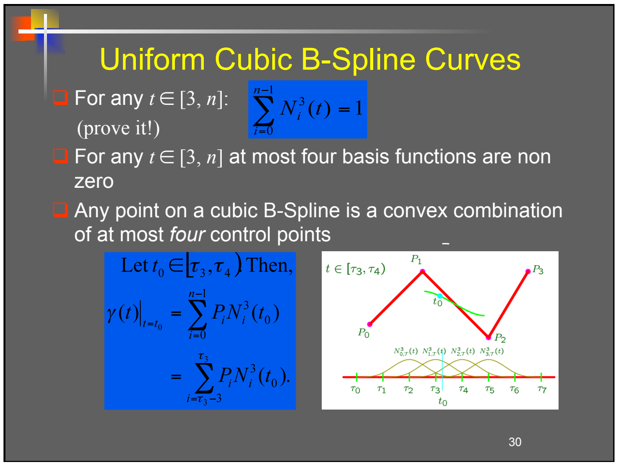
- Definition (uniform knot sequence, $\tau_i - \tau_{i-1} = 1$):

$$\gamma(t)=\sum_{i=0}^{n-1} P_i N_i^3(t), \quad t\in[3,n)$$

$$N_i^3(t)=\begin{cases} r^3/6 & r=t-\tau_i & t\in[\tau_i,\tau_{i+1}) \\ (-3r^3+3r^2+3r+1)/6 & r=t-\tau_{i+1} & t\in[\tau_{i+1},\tau_{i+2}) \\ (3r^3-6r^2+4)/6 & r=t-\tau_{i+2} & t\in[\tau_{i+2},\tau_{i+3}) \\ (1-r)^3/6 & r=t-\tau_{i+3} & t\in[\tau_{i+3},\tau_{i+4}) \end{cases}$$

$r \in [0,1]$

$N_i^3(t) = 0$ *elsewhere*



Uniform Cubic B-Spline Curves

- For any $t \in [3, n]$: $\sum_{i=0}^{n-1} N_i^3(t) = 1$ (prove it!)
- For any $t \in [3, n]$ at most four basis functions are non zero
- Any point on a cubic B-Spline is a convex combination of at most *four* control points

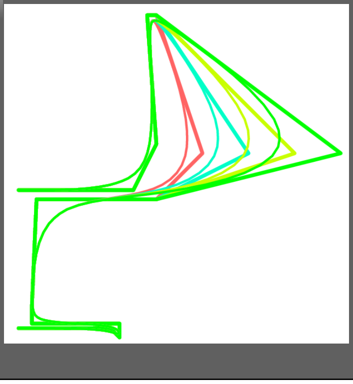
Let $t_0 \in [\tau_3,\tau_4)$ Then,

$$\gamma(t)\big|_{t=t_0} = \sum_{i=0}^{n-1} P_i N_i^3(t_0) = \sum_{i=\tau_3-3}^{\tau_3} P_i N_i^3(t_0).$$

Boundary Conditions for Cubic B-Spline Curves

- B-Splines do not interpolate control points
  - in particular, the uniform cubic B-spline curves do not interpolate the end points of the curve.
  - Why is the end points' interpolation important?
- Two ways are common to force endpoint interpolation:
  - Let $P_0 = P_1 = P_2$ (same for other end)
  - Add a new control point (same for other end) $P_{-1} = 2P_0 - P_1$ and a new basis function $N_{-1}^3(t)$.
- Question:
  - What is the shape of the curve at the end points if the first method is used?
  - What is the derivative vector of the curve at the end points if the first method is used?

Local Control of B-spline Curves



Control point P_i affects $\gamma(t)$ only for $t \in (\tau_i, \tau_{i+4})$

Properties of B-Spline Curves

- $$\gamma(t) = \sum_{i=0}^{n-1} P_i N_i^3(t), \quad t \in [3, n)$$
- For n control points, $\gamma(t)$ is a piecewise polynomial of degree 3, defined over $t \in [3, n)$

$$\gamma(t) \in \bigcup_{i=0}^{n-4} CH(P_i, \ldots, P_{i+3})$$

- $\gamma(t)$ is *affine invariant*
- $\gamma(t)$ follows the general shape of the control polygon and it is intuitive and ease to control its shape
- Questions:
  - What is $\gamma(\tau_i)$ equal to?
  - What is $\gamma'(\tau_i)$ equal to?
  - What is the continuity of $\gamma(t)$? Prove !

From Curves to Surfaces

- A curve is expressed as inner product of coefficients P_i and basis functions

$$C(u) = \sum_{i=0}^{n} P_i B_i(u)$$

(x_i, y_i, z_i)

- Treat surface as a *curves of curves*. Also known as *tensor product surfaces*
- Assume P_i is not constant, but are functions of a second, new parameter v:

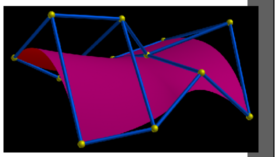
$$P_i(v) = \sum_{j=0}^{m} Q_{ij} B_j(v)$$

From Curves to Surfaces (cont'd)

$$C(u) = \sum_{i=0}^{n} P_i B_i(u)$$

$$= \sum_{i=0}^{n} \left(\sum_{j=0}^{m} Q_{ij} B_j(v) \right) B_i(u)$$

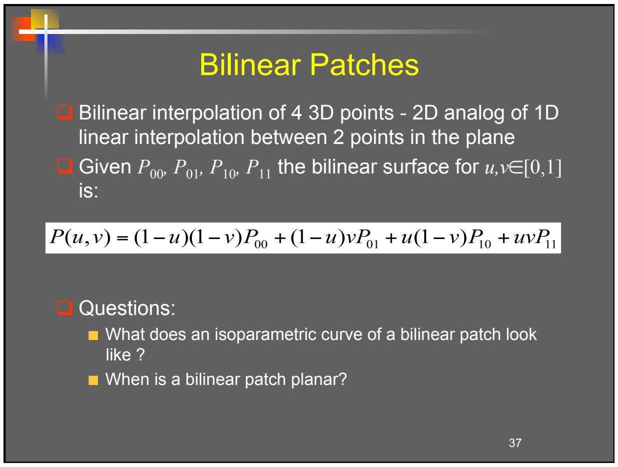
$$= \sum_{i=0}^{n} \sum_{j=0}^{m} Q_{ij} B_j(v) B_i(u)$$



$$S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} Q_{ij} B_j(v) B_i(u)$$

Surface Constructors

- Construction of the geometry is a first stage in any *image synthesis* process
- Use a set of high level, simple and intuitive, surface constructors:
  - Bilinear patch
  - Ruled surface
  - Boolean sum
  - Surface of Revolution
  - Extrusion surface
  - Surface from curves (skinning)
  - Swept surface

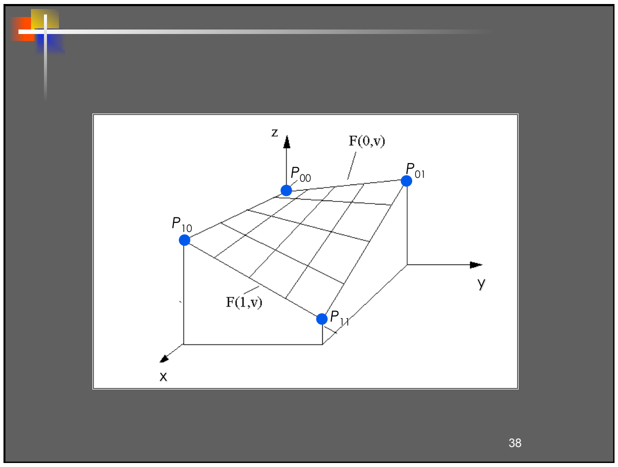


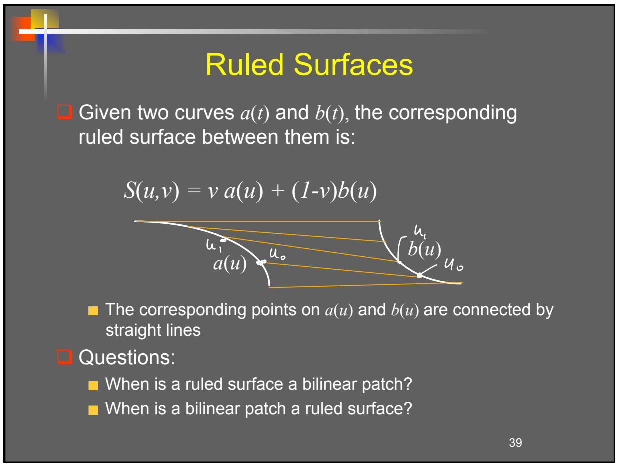
Bilinear Patches

- Bilinear interpolation of 4 3D points - 2D analog of 1D linear interpolation between 2 points in the plane
- Given P_{00}, P_{01}, P_{10}, P_{11} the bilinear surface for $u,v \in [0,1]$ is:

$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

- Questions:
  - What does an isoparametric curve of a bilinear patch look like ?
  - When is a bilinear patch planar?



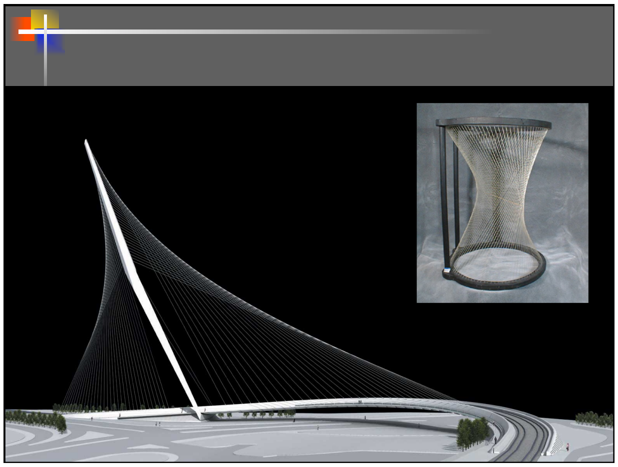


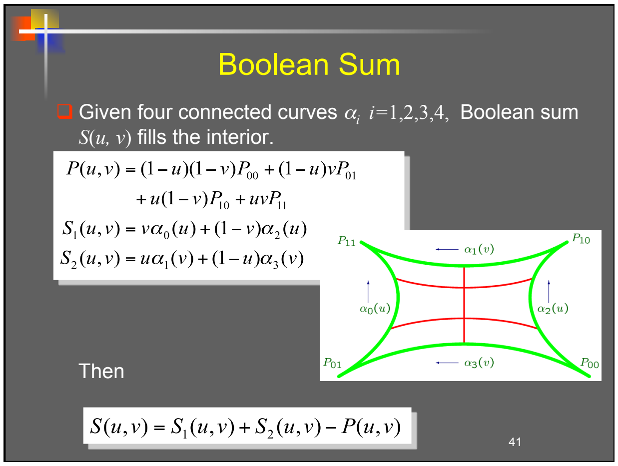
Ruled Surfaces

- Given two curves $a(t)$ and $b(t)$, the corresponding ruled surface between them is:

$$S(u,v) = v\,a(u) + (1-v)b(u)$$

  - The corresponding points on $a(u)$ and $b(u)$ are connected by straight lines
- Questions:
  - When is a ruled surface a bilinear patch?
  - When is a bilinear patch a ruled surface?





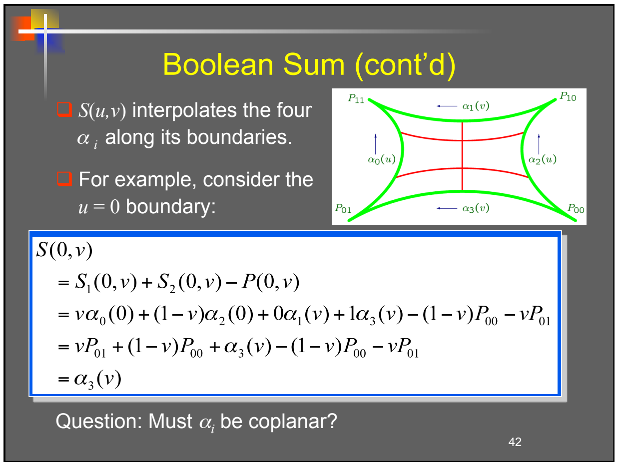
Boolean Sum

- Given four connected curves α_i $i=1,2,3,4$, Boolean sum $S(u, v)$ fills the interior.

$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$
$$S_1(u,v) = v\alpha_0(u) + (1-v)\alpha_2(u)$$
$$S_2(u,v) = u\alpha_1(v) + (1-u)\alpha_3(v)$$

Then

$$S(u,v) = S_1(u,v) + S_2(u,v) - P(u,v)$$

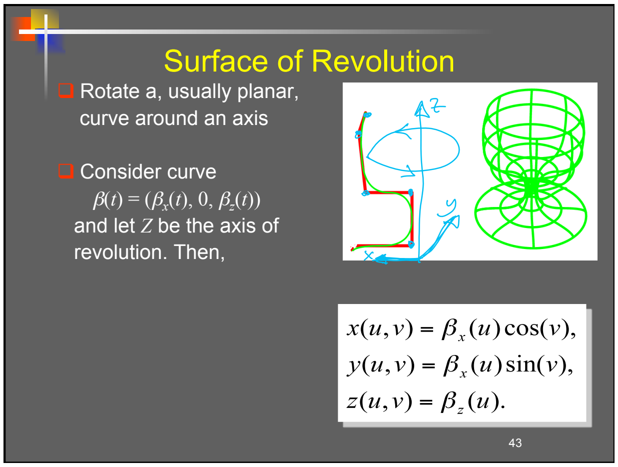


Boolean Sum (cont'd)

- $S(u,v)$ interpolates the four α_i along its boundaries.
- For example, consider the $u = 0$ boundary:

$$\begin{aligned} S(0,v) \\ &= S_1(0,v) + S_2(0,v) - P(0,v) \\ &= v\alpha_0(0) + (1-v)\alpha_2(0) + 0\alpha_1(v) + 1\alpha_3(v) - (1-v)P_{00} - vP_{01} \\ &= vP_{01} + (1-v)P_{00} + \alpha_3(v) - (1-v)P_{00} - vP_{01} \\ &= \alpha_3(v) \end{aligned}$$

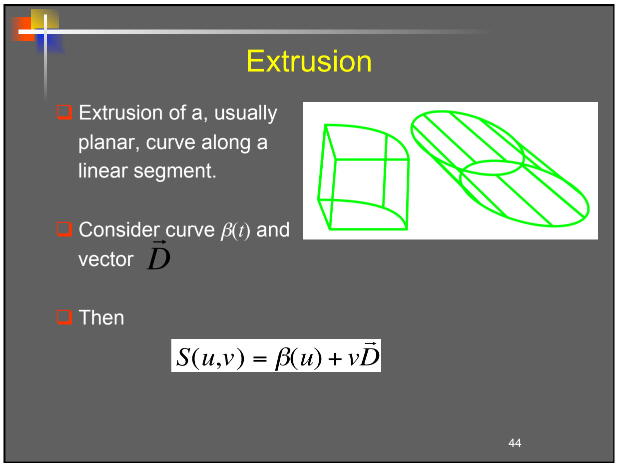
Question: Must α_i be coplanar?



Surface of Revolution

- Rotate a, usually planar, curve around an axis
- Consider curve $\beta(t) = (\beta_x(t), 0, \beta_z(t))$ and let Z be the axis of revolution. Then,

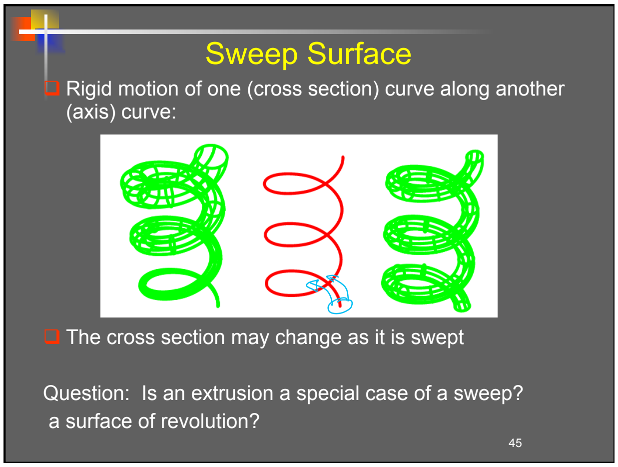
$$\begin{aligned} x(u,v) &= \beta_x(u)\cos(v), \\ y(u,v) &= \beta_x(u)\sin(v), \\ z(u,v) &= \beta_z(u). \end{aligned}$$



Extrusion

- Extrusion of a, usually planar, curve along a linear segment.
- Consider curve $\beta(t)$ and vector $\vec{D}$
- Then

$$S(u,v) = \beta(u) + v\vec{D}$$



Sweep Surface

- Rigid motion of one (cross section) curve along another (axis) curve:
- The cross section may change as it is swept

Question: Is an extrusion a special case of a sweep? a surface of revolution?