Geometric Hashing - on the whiteboard

Binary Space Partitions (BSP)

BSP and the painter algorithm

Quad trees and R-trees

BSP tree

Given a set of triangles $S = \{t_1 \dots t_n\}$ in 3D, a BSP T, for S is an tree where

1. Each leaf stores a triangle t_i
1. Each internal (non-leaf) node v stores a plane h_v and pointers to two children $v.right$, $v.left$
3. All triangles in the subtree $v.left$ are fully **below** h_v, and all triangles in $v.right$ are fully on or above h_v

See further example on the board.

Sometimes we need to split triangles to construct the BSP

If a (perfect) BSP exist, then for any location of a viewer, we can use the painter algorithm.

Numerous other applications in graphics. (e.g. combine with imposers/billboards)

If the number of triangles above and below h_v are roughly the same, then the height is $O(\log n)$

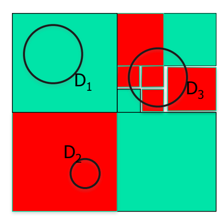
Quad Trees

A data simple data structure for geometric objects(e.g. points, houses, an image, 3D scene)

Support efficiently a very wide variety of queries.

Hierarchical Partition of the scene

QuadTrees



Assume we are given a red/green picture defined a $2^h \times 2^h$ grid. E.g. pixels.
Each pixel is either **green** or **red**.

(more general and interesting examples – soon)

Need to represent the shape "compactly"

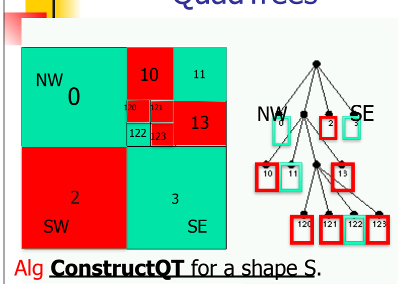
Need a data structure that could answers multiple types of queries. For example:

1. For a given point q, is q red or green ?

2. For a given query disk D, are there any green points in D ?

3. How many green points are there in D ?
4. Etc etc

QuadTrees



- Assume we are given a red/green picture defined on a $2^h \times 2^h$ grid of **pixels**.
- Each pixel has as a unique color (Green or Red)
- Every node $v \in \boldsymbol{T}$ is associated with a geometric region ***R(v)*** .
- This is the region that v *is* "in charge of".

Alg **ConstructQT** for a shape S.

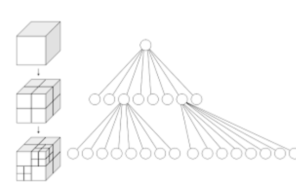
- **input** – a node $v \in \boldsymbol{T}$, and a shape *S.*
- **Output** – a Quadtree $\boldsymbol{T}_v$ representing the shape of *S within R(v)*).

- If *S* is fully green in *R(v)*, or S is fully red in *R(v)* – then
- *v* is a leaf, labeled Green or Red. Return ;
- Otherwise, divide *R(v)* into 4 equal-sized quadrants, corresponding to nodes v.*NW,* v.*NE,* v.*SW,* v.*SE*.
- Call **ConstructQT** recursively for each quadrant.

3D-Quadtess

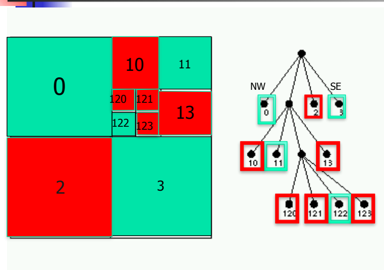
There are no 3D Quadtrees. We call them Oct-trees

Each node is either a leaf, or is split into 8 equal-volume octants.



credit for image: wikipedia.

QuadTrees



Consider a picture stored on an $2^h \times 2^h$ grid. Each pixel is either red or green.

We can represent the shape “compactly” using a QT.

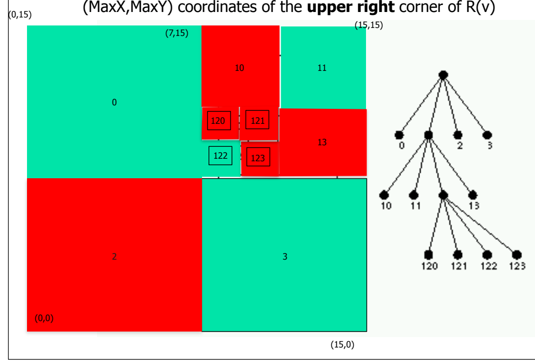
Height – at most h.
Point location operation – given a point q, is it black or white
– takes time O(h)
- could it be much smaller ?

Many other operations are very simple to implement.

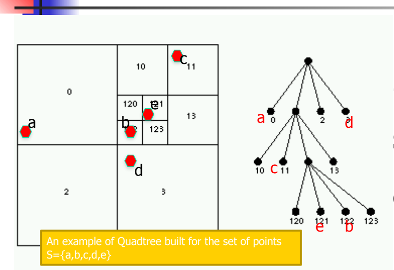
Storing the **range** R(v) of a node

Each node v is associated with a range R(v) – a square. The node v stores (in addition to other info) 4 values

(MinX,MinY) – coordinates of the **lower left** corner of R(v)
(MaxX,MaxY) coordinates of the **upper right** corner of R(v)



QuadTree for a set of points

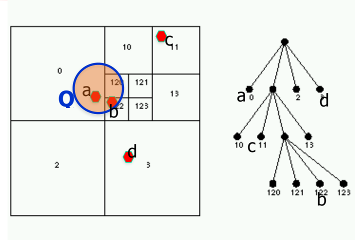


Now consider a set of points (red) but on a $2^h \times 2^h$ grid.

Splitting policy: Split until each quadrant contains ≤1 point.

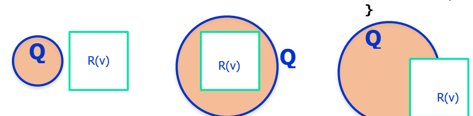
- Build a similar QT, but we stop splitting a quadrant when it contain ≤1 point (or some other small constant).
- Could be easily built by inserting the points one after the other. A leaf is split if contains 2 points.
- Point location operation – given a point q, is it black or white
- – takes time O(h) (and less in practice)
- Many other splitting polices are very simple to implement. (eg. a leaf could contain contains ≤17 points)

QuadTrees for a set of points



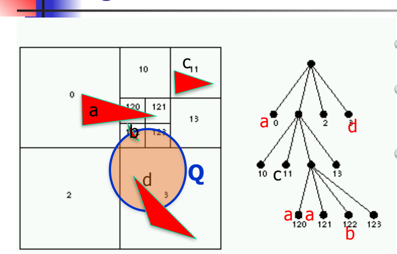
Report(Q,v) {
// Q – a query disk. v- node of a quad tree that stores a set S of points (e.g. S={a,b,c,s}).
Prints all data point the points in stored at the subtree rooted at v, which are also inside Q. */
1. If R(v) is disjoint from Q –**return** **//**no point to report at v's subtree
2. If R(v) is fully contained in Q – print all of S stores at the subtree rooted at v.
3. Else // *R*(*v*) partially overlaps Q. {
   - If v is a leaf – check each point in R(v) if inside Q and print if yes.
   - Else // v internal node
     - Report(Q, NW(v)) and ..
     - Report(Q, NE(v)) and ..
     - Report(Q, SW(v)) and ..
     - Report(Q, SE(v)) }

}



R(*v*) is disjoint from Q — *R*(*v*) is contained in Q — *R*(*v*) partially overlapQ

QuadTrees for shape



- Input: A set *S* of triangles $S=\{t_1 \ldots t_n\}$.
- Each leaf v stores a list *v.TriangleList* of all triangles intersecting *R(v)*.
- Splitting policy: Split a quadrant if it intersects more than 5 (say) triangle of *S*.

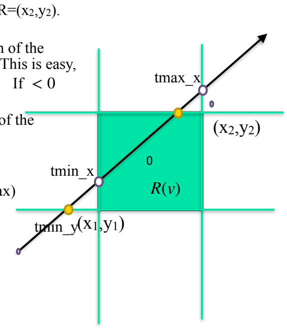
Note – a triangle might be stored in multiple leaves.
Some leaves might store no triangles.

Finding all triangles inside a query region Q. We essentially use the function *Report(Q,v)* from the previous slide (with minor modifications)

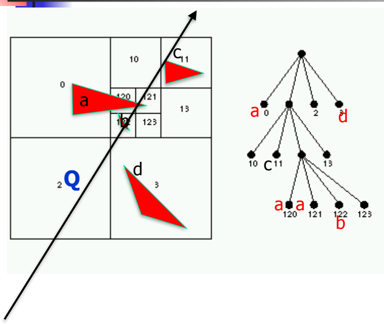
Ray tracing and QuadTrees

- Consider a quadrant *R*(*v*) with corners LL=(x_1,y_1) and UR=(x_2,y_2).
- To find if a ray $r = p_0 + t \cdot \vec{v}$ intersects this quadrant
  - Find tmin_x, tmax_x, where the ray is in the x-span of the quadrant (the vertical slab containing the quadrant). This is easy, since we only need to check the x-component of $\vec{v}$. If < 0 then this ray does not intersect *R*(*v*)
  - Find tmin_y, tmax_y, where the ray is in the y-span of the quadrant
  - Set tmin=max(tmin_x, tmin_y)
  - Set tmax=min(tmax_x, tmax_y)
  - The ray is inside the quadrant only for $t \in$ (tmin,tmax)

In 3D, we also check tmin_z, tmax_z



Ray tracing and QuadTrees

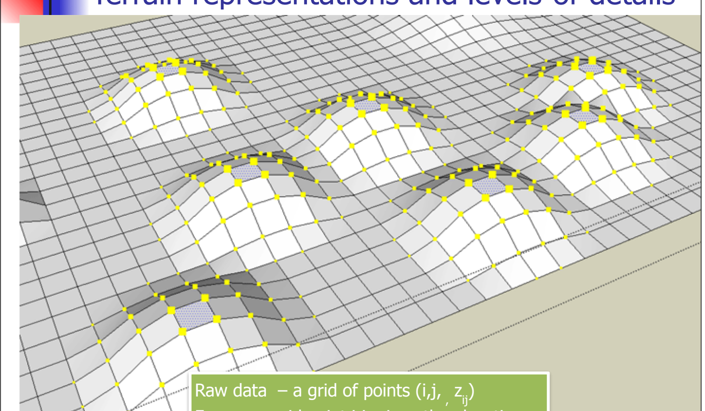


- Now, it is easy to find the first triangle hit by a ray r:
- Start from v=root. If empty, then continue tracing the ray from the point it leaves the quadrant.
- If v is internal node, check which of its quadrants is first hit by r, and continue recursively.
- If v=leaf, check each triangle in v

Inserting a new triangle

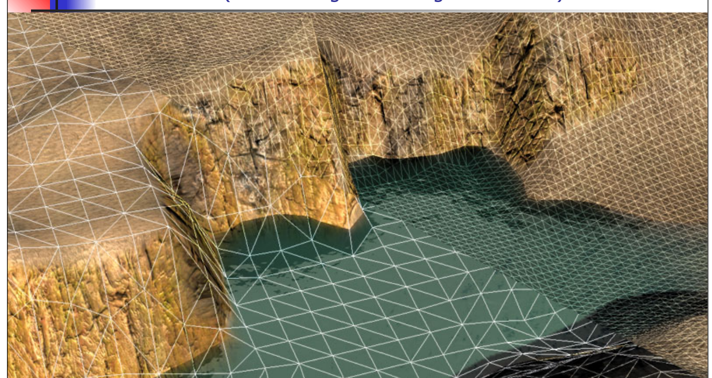
```
insert(triangle ti ,node v) {
    // Inserting a new triangle ti into an existing node v of the Quadtree.
    // v is not necessarily a leaf.
    If v is NULL - Error
    If R(v) is disjoint from ti  (share no points)– Return.  Nothing to do.
    If v is not a leaf,  then for each child u of v, call insert(ti,u);
    Else // v is a leaf
    Add ti to v.TrianglesList
    If number of triangles in v.SegmentsList  is too long (e.g. >5) Call Split(v)
}
-----------------
Split(v){
    // Assumption – v is a leaf, but has too many triangles in its list.
    // Create 4 children for v (make sure they know which regions they cover.)
    For each child u of v
        For each segment s in v.TrianglesList Call insert(s, u)
    Empty v.TrianglesList
}
```

Terrain representations and levels-of-details



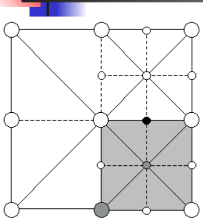
Raw data – a grid of points (i,j, z_{ij})
For every grid point i,j, given the elevation z_{ij}

(TIN – Triangulated Irregular Network)



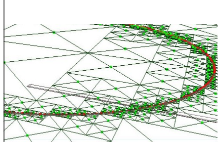
Each triangle approximately fits the surface below it

How to find good triangulation ?



- Input – a very large set of points $S=\{ (i,j, z_{ij}) \}$.
- z_{ij} is the elevation at point ***(i,j) (latitude and longitude)***
- Want to create a surface, consists of triangles, where each triangle interpolates the data points underneath it.
- Idea: Build a QT *T* for the 2D points.
- (If want triangles: Each quadrant is split into 2 right-hand triangles)
- Assign to each vertex the height of the terrain above it.
- The approximated elevation of the terrain at any point *(x,y)* is the linear interpolation of its elevated vertices.

QT Split Policy: Splitting a quadrant into 4 sub-quadrants:



- split a node ***v*** if for some date point $(x_i , y_i)\in R(v)$, the elevation of z_{ij} is too far from the the corresponding triangle. If not, leave ***v*** as a leaf.
- That is, for any point *(i,j)* on the plane, the elevation (i,j , z_{ij}) it is too far from the interpolated elevation.
- Note: A quadrant might contain a huge number of points, but they behave smoothly. E.g. all a the sloop of a mountain, but this slope is more or less linear.

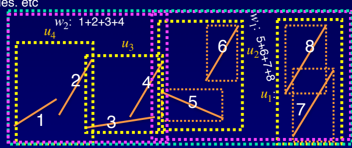
Level Of Details

- Idea – the same object is stored several times, but with a different level of details
- Coarser representations for distant objects
- Decision which level to use is accepted `on the fly'
  (eg in graphics applications, if we are far away from a terrain, we could tolerate usually large error. E.g., sub pixels error are not noticeable.)

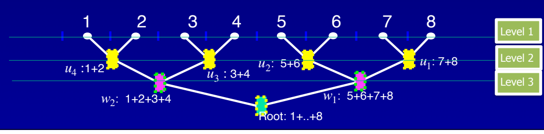
69,451 polys | 2,502 polys | 251 polys | 76 polys

R-trees

- Input: A set S of shapes (segments in this example. Triangles in graphics apps)
- Build a tree that could expedite
  - (i) finding the segments intersecting a query region,
  - (ii) answering ray tracing
  - (iii) Emptiness queries. etc



- We compute for each segment its bounding box (rectangle).
- These are the leaves of *T*. Call them ``Level 1".
- Find the nearest pair of segments (say 7,8). Remove them from level 1, and replace them by a single BB encapsulate both. It corresponds to a node of level 2.
- Repeat until no vertex is left in level 1.
- Next, pick the nearest two BBs from level 2, and replace them by a vertex at level 3.
- In general, each internal node *v* in level **j** is created by merging two children nodes of level **j-1**.
  - $BB(v) = BB\left(BB(v.right) \bigcup BB(v.left)\right)$
- Repeat until we are left with one bounding box.



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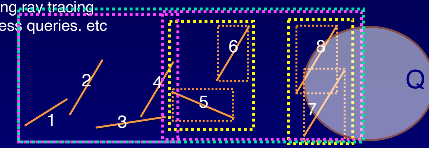
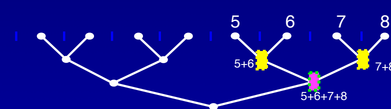
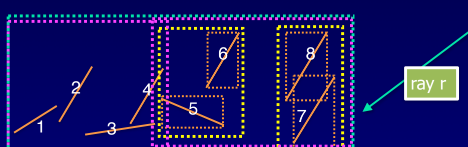


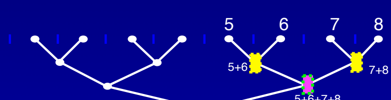
Once a query region Q is given, we need to report the segments intersecting Q.
Check if Q intersects BB(root)
If not, we are done. If yes, check recursively if Q intersects BB(v.left) and BB(v.right)



R-trees

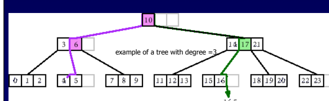


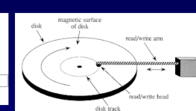
Analogously for a query ray *r*

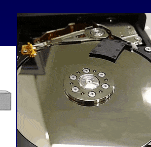


R-trees in practice. Memory Hierarchy, and advantages of multiple children Large degree helps

- In practice, it is sometimes preferable to create trees with a very large degrees (instead of binary trees).
- For example, each internal node, will have between 100 to 500 children



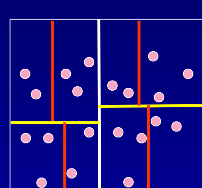




- Consider a very simplistic model of the computer memory - fast main memory, and slow secondary memory. (your computer follows this model, probably with more than 2 types of memory, and probably SSD instead of disks, but this model still applies.
- Only small portion of the tree could be stored in the main memory.
- Consider point location operation (find the segment containing a query point)
- We start the search by visiting the root, then one of its children, one of its grand-children … until we reach a leaf.
- The seek-time in disks, and even in SSD, is much slower than the seek-time for main memory. Therefor, once the head of the disks is located in the correct place, we usually read a bucket - about 4KByte of memory.
- The bottleneck of the *search/insert/delete* operation is the number of seek operations (number of I/Os).
- The number of seek-operation is proportional to height of the tree.
- Say $n = 10^9$. The height of a tree of degree 2 with n leaves is $\log_2(10^9) \approx 30$, so 30 seek operations are needed.
- If each node contains about 1000 segments, or keys, then the height (and number of I/Os) is only $\log_{1000}(10^9) = 3$
- The root and possibly its children are always in main memory, so this number of only 1 or 2.
- R-trees are the most popular and important data structures for very large spatial data.
- If the stored items are 1-dimensional (rather than multi-dim), then B-trees are used instead of R-trees. They are very convent for insertion/deletion and other operations.

2D-Trees (and in higher dimension, kD-trees)

- Given a set of points in 2D.
- Bound the points by a rectangle.
- Split the points into two (almost) equal size groups, using a horizontal line, or vertical line. (first horizontal, then vertical, back to horizontal etc)
- (in $\mathbb{R}^3$, split by a plane orthogonal to the
  - $x-$axis,
  - then orthogonal to $y-$axis,
  - then $z-$axis,
  - and back to $x - axis$ etc
- Continue recursively to partition the subsets, until they are small enough.



2D-Tree

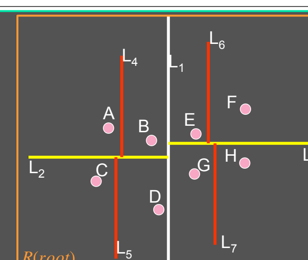
- Partitions 2D space into axis-aligned rectangular regions.
- Nodes stores partition lines. However each node v corresponds to a region $R(v)$ in the plane. (the reasons that nodes don't need to store $R(v)$ is that $R(v)$ could be computed from by the path from the root to $v$and leaves represent input points.

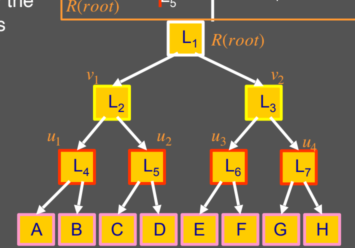
construction complexity:

$$T(n) = \begin{cases} O(1) & n = 1 \\ O(n) + 2T\left(\frac{n}{2}\right) & n > 1 \end{cases}$$

$$T(n) = O(n \log n)$$

- Height: $O(\log n)$





We saw a family of hierarchical trees

Def: Hirarchical tree tree T : $R(v) \subseteq R(parent(v))$ for every non-root noded v,
Where $R(v)$ is the region of node v

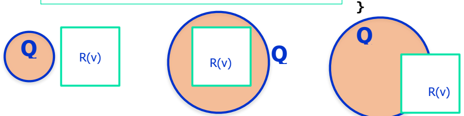
Example, Quadtree, R-tree, kD-tree

Augmenting the tree: We can store at each internal node of v of T additional information that relates to data in the subtree of v. For example:
1. The number of data points in its v's subtree
2. Max,
3. Min,
4. sum-of-RGB
5. etc

Report(Q,v) {
// Q – a query disk. v- node of a quad tree that stores a set S of points (e.g. S={a,b,c,s}).
Prints all data point the points in stored at the subtree rooted at v, which are also inside Q. */

1. If R(v) is disjoint from Q –**return** //no point to report at v's subtree
2. If R(v) is fully contained in Q – print all of S stores at the subtree rooted at v.
3. Else // $R(v)$ partially overlaps Q. {
   - If v is a leaf – check each point in R(v) if inside Q and print if yes.
   - Else // v internal node
     - Report(Q, NW(v)) and ..
     - Report(Q, NE(v)) and ..
     - Report(Q, SW(v)) and ..
     - Report(Q, SE(v)) }

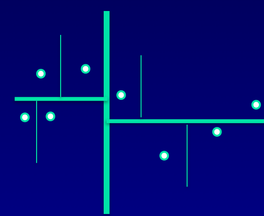
}



$R(v)$ is disjoint from Q $R(v)$ is contained in Q $R(v)$ partially overlapQ

Question: Which values should we maintain so we could find the average efficiently

A word about theoretical guaranties



- All these trees are very efficient for realistic data and queries. Most regions in the tree are either fully inside the query or fully outside
- All works well in 3D and 4D

- As far as theoretical guaranties goes, bounds are less striking Every thouhWe can prove: In a kD-tree, a query with axis-parallel query region visits at most $O(\sqrt{n})$ (in 2D) and $O(n^{2/3})$ (in 3D)