Triangulations

- A triangulation of set of points in the plane is a partition of the convex hull to triangles whose vertices are the points, and are empty of other points.
- There are an exponential number of triangulations of a point set.

Piecewise Linear Interpolation

- The height of a point $P$ inside a triangle is determined by the height of the triangle vertices, and the location of $P$.
- The result depends on the triangulation.

An $O(n^2)$ Triangulation Algorithm

- Repeat until impossible:
  - Select two sites.
  - If the edge connecting them does not intersect previous edges, keep it.

Motivation

- Assume a height value is associated with each point.
- A triangulation of the points defines a piecewise-linear surface of triangular patches.

Computational Geometry

Chapter 8

Delaunay Triangulation

Barycentric Coordinates and linear interpolation

- Any point inside a triangle can be expressed uniquely as a convex combination of the triangle vertices:

$$ f(p) = \alpha_1 f(v_1) + \alpha_2 f(v_2) + \alpha_3 f(v_3) $$
"Quality" Triangulations

- Let $\alpha(T) = (\alpha_1, \alpha_2, ..., \alpha_d)$ be the vector of angles in the triangulation $T$ in increasing order.
- A triangulation $T_1$ will be "better" than $T_2$ if $\alpha(T_1) > \alpha(T_2)$ lexicographically.
- The Delaunay triangulation is defined as the "best" triangulation.

The Number of Triangles

- The number of triangles in a triangulation of $n$ points depends on the number of vertices $h$ on the convex hull.

$$ t = 2n - 2h - 2 $$

Proof: We count the sum of angles of all triangles in two ways:

First way: $180^\circ t$ (every triangle contributes $180^\circ$ degrees).
Second way: The same number equals: $360^\circ(n - h) + 180^\circ(h - 2)$

Improving a Triangulation

- In any convex quadrangle, an edge flip is possible. If this flip improves the triangulation locally, it also improves the global triangulation.

- If an edge flip improves the triangulation, the first edge is called illegal.
  - "Improves" means that that vector of angles is lexicographic larger.

Thales' Theorem

- Let $C$ be a circle, and $l$ a line intersecting $C$ at points $a$ and $b$. Let $p, q, r$ and $s$ be points lying on the same side of $l$, where $p$ and $q$ are on $C$, $r$ inside $C$ and $s$ outside $C$. Then:

Illegal Edges - cont

- Lemma: An edge $pq$ is illegal iff one of its opposite vertices is inside the circle defined by the other three vertices.

- Theorem: A Delaunay triangulation does not contain illegal edges.
- Corollary: A triangle is Delaunay iff the circle through its vertices is empty of other sites (the empty-circle condition).
- Note: The Delaunay triangulation might not be unique if more than three sites are co-circular.

Illegal Edges

- Lemma: An edge $pq$ is illegal iff one of its opposite vertices is inside the circle defined by the other three vertices.
- Proof: By Thales' theorem.
Naïve Delaunay Algorithm
- Start with an arbitrary triangulation. Flip any illegal edge until no more exist.
- Requires proof that there are no local minima.
- Could take a long time to terminate.

O(n^2) Delaunay Triangulation Algorithm
- Repeat until impossible:
  - Select a triple of sites.
  - If the circle through them is empty of other sites, keep the triangle whose vertices are the triple.

The In-Circle Test
**Theorem:** If a, b, c, d form a CCW convex polygon, then d lies in the circle determined by a, b and c if:

$$ x^2 + y^2 = (a_x - b_x)^2 + (a_y - b_y)^2 $$
$$ x^2 + y^2 = (b_x - c_x)^2 + (b_y - c_y)^2 $$
$$ x^2 + y^2 = (c_x - d_x)^2 + (c_y - d_y)^2 $$

**Proof:** We prove that equality holds if the points are co-circular. There exists a center q and radius r such that:

and similarly for b, c, d:

So these four vectors are linearly dependent, hence their det vanishes.

**Corollary:** d \in circle (a, b, c) iff b \in circle (c, d, a) iff c \notin circle (d, a, b) iff a \notin circle (b, c, d)

Delaunay Triangulation by Duality
**Correctness**: Can these triangles intersect?
- Case A: a cite lies inside a triangle:
  - Impossible, because the circle is empty, hence also the triangle p.p.p.
- Case B: Two triangulation edges intersect, but no cite-inside-a-tri.
  - Each yellow edge must intersect a white one \( \Rightarrow \) two white edges also intersect, but these edges are in disjoint Voronoi cells. Contradiction.
- Case C is possible.

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Delaunay Triangulation: Main Property

Let \( S \) be a set of points in the plane. Then,

(i) \( p_ip jp k \in S \) are vertices of a triangle (face) of \( DT(S) \)
\[\Leftrightarrow\] The circle passing through \( p_ip jp k \) is empty;

(ii) \( \overline{p_ip j} \) (for \( p_ip j \in S \)) is an edge of \( DT(S) \)
\[\Leftrightarrow\] There exists an empty circle passing through \( p_ip j \).

Proof: Dualize the Voronoi-diagram theorem.

Corollary:
A triangulation \( T(S) \) is \( DT(S) \)
\[\Leftrightarrow\] Every circumscribing circle of a triangle \( \Delta \in T(S) \) is empty.

Algorithm Complexity

- Point location for every point: \( O(\log n) \) time.
- Flips: \( \Theta(n) \) expected time in total (for all steps).
- Total expected time: \( O(n \log n) \).
- Space: \( \Theta(n) \).

Number of Flips

- Theorem: The expected number of edges flips made in the course of the algorithm (some of which also disappear later) is at most 6n.
- Proof:
During insertion of vertex \( p_i \), \( k_i \) new edges are created:
3 new initial edges, and \( k_i-3 \) due to flips.
Backward analysis: \( E[k] = \) the expected degree of \( p_i \) after the insertion is complete = 6 (Euler).
The Delaunay Triangulation and Convex Hulls

- Euclidean Minimum Spanning Tree (EMST): A tree of minimum length connecting all the sites.
- Relative Neighborhood Graph: Two sites $p, q$ are connected if
- Gabriel Graph (GG): Two sites $p, q$ are connected iff the circle whose diameter is $pq$ is empty of other sites.

**Theorem:** $\text{EMST} \subseteq \text{RNG} \subseteq \text{GG} \subseteq \text{DT}$

**Proof**

- The intersection of a plane with the paraboloid is an ellipse whose projection to the plane is a circle.
- $s$ lies within the circumcircle of $p, q$ iff $s'$ lies on the lower side of the plane passing through $p', q', r'$.
- $p, q, r \in S$ form a Delaunay triangle iff $p', q', r'$ form a face of the convex hull of $S$. 

The Voronoi Diagram and Convex Hulls

- Given a set $S$ of points in the plane, associate with each point $p \in S$ the plane tangent to the paraboloid at $p$: $z = 2ax + 2by - (a^2 + b^2)$.
- $\text{VD}(S)$ is the projection to the $(x, y)$ plane of the 1-skeleton of the convex polyhedron formed from the intersection of the halfspaces above these planes.