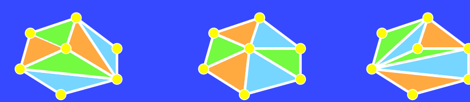
Computational Geometry

Chapter 8

Delaunay Triangulation

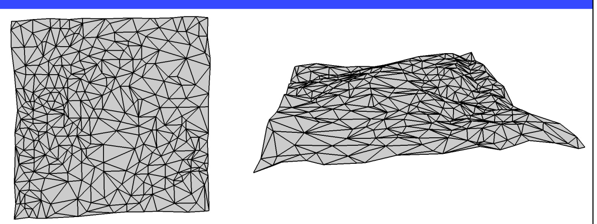
Triangulations

- A *triangulation* of set of points in the plane is a *partition* of the convex hull to triangles whose vertices are the points, and are empty of other points.
- There are an exponential number of triangulations of a point set.



Motivation

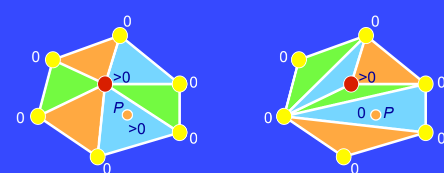
- Assume a height value is associated with each point.
- A triangulation of the points defines a *piecewise-linear* surface of triangular patches.



2D 3D

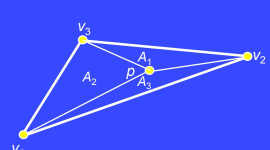
Piecewise Linear Interpolation

- The height of a point *P* inside a triangle is determined by the height of the triangle vertices, and the location of *P*.
- The result depends on the triangulation.



Barycentric Coordinates

- Any point inside a triangle can be expressed *uniquely* as a *convex* combination of the triangle vertices:



$$p = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$\alpha_i = \frac{A_i}{A_1 + A_2 + A_3}$$

$$\alpha_i \geq 0,\ \alpha_1 + \alpha_2 + \alpha_3 = 1$$

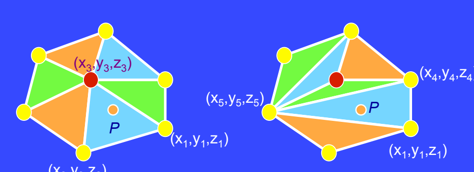
Piecewise Linear Interpolation

$$x_P = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

$$y_P = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3$$

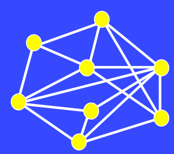
$$\Downarrow$$

$$z_P = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3$$



An O(n^3) Triangulation Algorithm

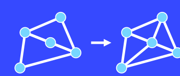
- Repeat until impossible:
  - Select two sites.
  - If the edge connecting them does not intersect previous edges, keep it.

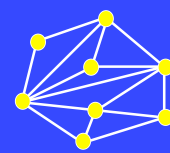


An O($n$log$n$) Triangulation Algorithm

- Construct the convex hull, and connect one arbitrary vertex to all others.
- Insert the other sites one after the other.
- Two possibilities:
  - Inside a triangle (one triangle becomes three).

  - On an edge (two triangles become four)

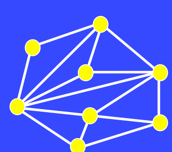




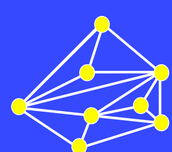
The Number of Triangles

- The number of triangles in a triangulation of n points depends on the number of vertices h on the convex hull.

$t = 2n-h-2$



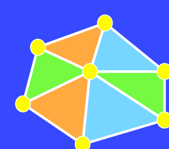
$n = 8$



$h = 6 \rightarrow t = 8$ $h = 5 \rightarrow t = 9$

"Quality" Triangulations

- Let $\alpha(T) = (\alpha_1, \alpha_2 ,.., \alpha_{3t})$ be the vector of angles in the triangulation T in increasing order.
- A triangulation T_1 will be "better" than T_2 if $\alpha(T_1) > \alpha(T_2)$ lexicographically.
- The Delaunay triangulation is the "best".



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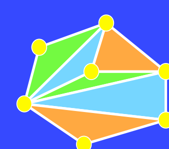

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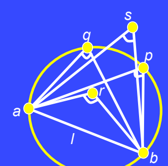
Thales' Theorem

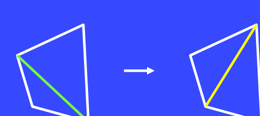
- Let C be a circle, and l a line intersecting C at points a and b. Let p, q, r and s be points lying on the same side of l, where p and q are on C, r inside C and s outside C. Then:

$$\angle arb > \angle apb = \angle aqb > \angle asb$$



Improving a Triangulation

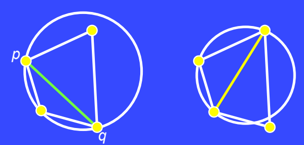
- In any convex quadrangle, an *edge flip* is possible. If this flip *improves* the triangulation locally, it also improves the global triangulation.



- If an edge flip improves the triangulation, the first edge is called *illegal*.

Illegal Edges

- **Lemma:** An edge *pq* is illegal iff one of its opposite vertices is inside the circle defined by the other three vertices.
- **Proof:** By Thales' theorem.



- **Theorem:** A Delaunay triangulation does not contain illegal edges.
- **Corollary:** A triangle is Delaunay iff the circle through its vertices is empty of other sites (the *empty-circle* condition).
- **Corollary:** The Delaunay triangulation is not unique if more than three sites are co-circular.

$O(n^4)$ Delaunay Triangulation Algorithm

- Repeat until impossible:
  - Select a triple of sites.
  - If the circle through them is empty of other sites, keep the triangle whose vertices are the triple.

The In-Circle Test

Theorem: If *a*,*b*,*c*,*d* form a CCW convex polygon, then *d* lies in the circle determined by *a*, *b* and *c* iff:

$$\det\begin{pmatrix} a_x & a_y & a_x^2+a_y^2 & 1 \\ b_x & b_y & b_x^2+b_y^2 & 1 \\ c_x & c_y & c_x^2+c_y^2 & 1 \\ d_x & d_y & d_x^2+d_y^2 & 1 \end{pmatrix} > 0$$

Proof: We prove that equality holds if the points are co-circular. There exists a center *q* and radius *r* such that:

$$(a_x-q_x)^2+(a_y-q_y)^2=r^2$$

and similarly for *b*, *c*, *d*:

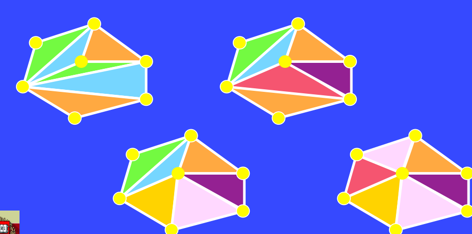
$$\begin{pmatrix} a_x^2+a_y^2 \\ b_x^2+b_y^2 \\ c_x^2+c_y^2 \\ d_x^2+d_y^2 \end{pmatrix} - 2q_x \begin{pmatrix} a_x \\ b_x \\ c_x \\ d_x \end{pmatrix} - 2q_y \begin{pmatrix} a_y \\ b_y \\ c_y \\ d_y \end{pmatrix} + (q_x^2+q_y^2-r^2) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

So these four vectors are linearly dependent, hence their det vanishes.

Corollary: $d\in$circle(*a*,*b*,*c*) iff $b\in$circle(*c*,*d*,*a*) iff $c\notin$circle(*d*,*a*,*b*) iff $a\notin$circle(*b*,*c*,*d*)

Naïve Delaunay Algorithm

- Start with an arbitrary triangulation. Flip any illegal edge until no more exist.
- Requires proof that there are no local minima.
- Could take a long time to terminate.



Delaunay Triangulation by Duality

- General position assumption: There are no four co-circular points.
- Draw the dual to the Voronoi diagram by connecting each two neighboring sites in the Voronoi diagram.
- **Corollary:** The DT may be constructed in O(*n*log*n*) time.

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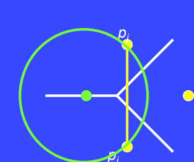
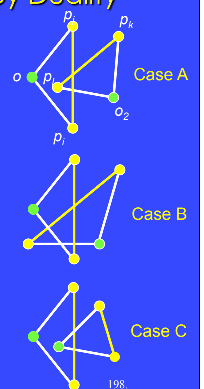
Delaunay Triangulation by Duality – Correctness

- It is easy to see that any resulting triangle is Delaunay, but can these triangles intersect ?
- Let *S* be a set of sites, and let DT(*S*) be the dual of the Voronoi diagram.
- Assume by contradiction that p_ip_j intersects another edge in DT(*S*).
- p_i and p_j are neighbors, hence there exists an empty circle through them centered on their bisector.



Delaunay Triangulation by Duality – Correctness

- Case A:
  - Impossible, because the circle is empty, hence also the triangle $p_i p_j o$.
- Case B:
  - Each yellow edge must intersect a white one ⇒ two white edges also intersect, but these edges are in disjoint Voronoi cells. Contradiction.
- Case C is possible.



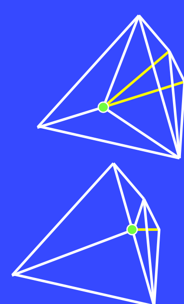
Delaunay Triangulation: Main Property

- **Theorem**:
  Let S be a set of points in the plane. Then,
  (i) $p_i,p_j,p_k \in S$ are vertices of a triangle (face) of DT(S)
  ↔ The circle passing through p_i,p_j,p_k is empty;
  (ii) $\overline{p_i p_j}$ (for $p_i,p_j \in S$) is an edge of DT(S)
  ↔ There exists an empty circle passing through p_i,p_j.
- **Proof**: Dualize the Voronoi-diagram theorem.

- Corollary:
  A triangulation T(S) is DT(S)
  ↔ Every circumscribing circle of a triangle $\Delta \in$ T(S) is empty.

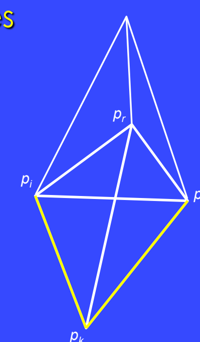
On O($n$log$n$) Delaunay Triangulation Algorithm

- **Randomized incremental algorithm:**
- Form bounding triangle which encloses all the sites.
- Add the sites one after another in random order and update triangulation.
- If the site is inside an existing triangle:
  - Connect site to triangle vertices.
  - Check if a 'flip' can be performed on one of the triangle edges. If so – check recursively the neighboring edges.
- If the site is on an existing edge:
  - Replace edge with four new edges.
  - Check if a 'flip' can be performed on one of the opposite edges. If so – check recursively the neighboring edges.

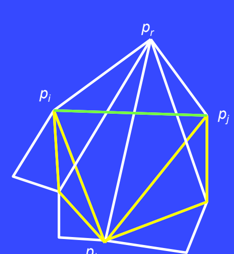


Flipping Edges

- A new vertex p_r is added, causing the creation of the edges $p_i p_r$ and $p_j p_r$.
- The legality of the edge $p_i p_j$ (with opposite vertex) p_k is checked.
- If $p_i p_j$ is illegal, perform a flip, and recursively check edges $p_i p_k$ and $p_j p_k$, the new edges opposite p_r.
- Notice that the recursive call for $p_i p_k$ cannot eliminate the edge $p_r p_k$.
- **Note:** All edge flips replace edges opposite the new vertex by edges incident to it!



Flipping Edges - Example



Number of Flips

- **Theorem:** The expected number of edges flips made in the course of the algorithm (some of which also disappear later) is at most 6n.
- **Proof:**
  During insertion of vertex p_i, k_i new edges are created: 3 new initial edges, and k_i-3 due to flips.
  Backward analysis: E[k_i] = the expected degree of p_i after the insertion is complete = 6 (Euler).

Algorithm Complexity

- Point location for every point: $O(\log n)$ time.
- Flips: $\Theta(n)$ expected time in total (for all steps).
- Total expected time: $O(n \log n)$.
- Space: $\Theta(n)$.

Relatives of the Delaunay Triangulation

- **Euclidean Minimum Spanning Tree (EMST):** A tree of minimum length connecting all the sites.
- **Relative Neighborhood Graph:** Two sites *p*, *q* are connected iff

$$d(p,q) \le \min_{r \in P, r \ne p,q} \max(d(p,r), d(q,r))$$

- **Gabriel Graph (GG):** Two sites *p*, *q* are connected iff the circle whose *diameter* is *pq* is empty of other sites.

Theorem: EMST ⊆ RNG ⊆ GG ⊆ DT

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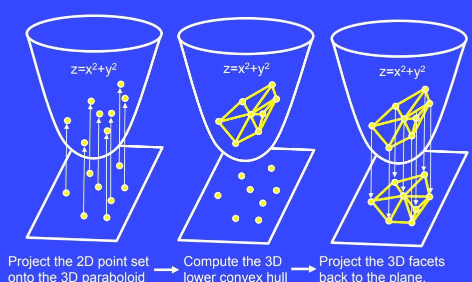
The Delaunay Triangulation and Convex Hulls



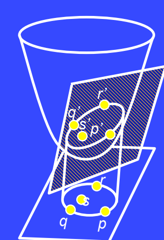
The 2D triangulation is Delaunay !

Proof

- The intersection of a plane with the paraboloid is an ellipse whose projection to the plane is a circle.
- *s* lies within the circumcircle of *p*, *q*, *r* iff *s*' lies on the lower side of the plane passing through *p*', *q*', *r*'.

↓

- *p*, *q*, *r* ∈ *S* form a Delaunay triangle iff *p*', *q*', *r*' form a face of the convex hull of *S*'.



The Voronoi Diagram and Convex Hulls

- Given a set *S* of points in the plane, associate with each point $p=(a,b) \in S$ the plane tangent to the paraboloid at *p*:
  $z = 2ax+2by-(a^2+b^2)$.
- VD(*S*) is the projection to the (x,y) plane of the 1-skeleton of the convex polyhedron formed from the intersection of the halfspaces above these planes.