Dating: Who wins the battle of the sexes?
Stable marriage (matching) algorithm.

There are $n$ males and $n$ females
Each female has her own ranked preference list of all the males
  - E.g., women #1 most prefers male #3 over any other male.
Each male has his own ranked preference list of the females
How should we match them (1-to-1)

1. $3,2,5,1,4$
2. $1,2,5,3,4$
3. $4,3,2,1,5$
4. $1,3,4,2,5$
5. $1,2,4,5,3$

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2. $5,2,1,4,3$
3. $4,3,5,1,2$
4. $1,2,3,4,5$
5. $2,3,4,1,5$
**Rogue Couples**

- Consider a given matching $M$. Now suppose that some pair (male, female) who are not married to each other, actually prefer each other over their partners.

- They will be called a rogue couple.
- They both would gain from dumping their mates and marrying each other.

- A matching is called stable if it does not contain any rogue couples.

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The study of stability will be the subject of the entire lecture.

*We will: Analyze various mathematical properties of an algorithm that looks a lot like 1950’s dating.*

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Given a set of preference lists, how do we find a stable pairing?

*Wait! We don’t even know that such a pairing always exists!*
Is there always a stable matching?

- Will show: every set of preference lists have a stable pairing.
- Will prove it by presenting a fast algorithm that, given any set of input lists, will output a stable pairing.

Terminology and principles

- A male can propose (marriage) to a female.
- A female can reject the proposal.
- During most of the process, a female would not accept a proposal, but would tell a proposing male “maybe”.
- This is called "putting the male on a string”
- Once a male is rejected, he crosses off from his list the rejecting female – he will not propose to her again.
- Once a male proposes, he cannot change his mind until he is rejected.

The Traditional Marriage Algorithm
Traditional Marriage Algorithm (TMA)

1) repeat
   = Morning
     • Each male to the best female whom he has not yet crossed off
   = Afternoon (for each females with at least one proposal)
     • To today’s best offer: “Maybe, come back tomorrow” (putting him on a string)
     • All other proposals are rejected.
   = Evening
     • Any rejected male crosses the rejecting female off his list.
   )Until all males are on strings.

2) Each female marries the last male she just said “maybe”

Note: Each male proposes to females in decreasing order on his list.

Lemma: If a female has a male b on a string, then she will either marry him, or marry someone she prefers over him.

Proof:
   - She would only let go of b in order to “maybe” b’ which she prefers over b
   - She would only let go of b’ for someone b’’ she prefers over b’ etc.

When the process terminates, she is left with someone she prefers over b.

Corollary: Each female will marry her absolute favorite of the males who visit her during the Traditional Marriage Algorithm (TMA)
Lemma: No male can be rejected by all the females

- Proof by contradiction.
- Suppose male $b$ is rejected by all the females. At that point:
  - Each female must have a suitor other than $b$
    (By previous Lemma, once a female has a suitor she will always have at least one)
  - The $n$ females have $n$ suitors, $b$ not among them.
    Thus, there are at least $n+1$ males.
Contradiction

Theorem: The TMA always terminates after at most $n^2$ days

Proof
- The total length of the lists of all males is $n \times n = n^2$.
- Each day at least one male gets a “No”, so at least one female is deleted from one of the lists.
- Therefore, the number of days is bounded by the original size of the master list $= n^2$.

Great! We know that TMA will terminate and produce a pairing.

But is it stable?
Theorem: TMA. Produces a stable pairing.
1. Let \( m_1 \) and \( f_1 \) be any couple in \( T \).
2. Suppose \( m_1 \) prefers \( f_2 \) over \( f_1 \).
3. We will argue that \( f_2 \) prefers her husband over \( m_1 \).
4. During TMA, male \( m_1 \) proposed to \( f_2 \) before he proposed to \( f_1 \).
5. Hence, at some point \( f_2 \) rejected \( m_1 \) for someone she preferred.
6. By the Improvement lemma, the man she married was also preferable to \( m_1 \).
7. Thus, every male will be rejected by any female he prefers to his wife.
8. \( T \) is stable. QED.

Forget TMA for a moment

• How should we define what we mean when we say “the optimal female for male \( b \)”?
  
  Flawed Attempt:
  “The female at the top of \( b \)’s list”

The Optimal female

• A male’s optimal female is the highest ranked female for whom there is some stable matching in which they are married.
• (note – this is not always the highest female on his list).
• She is the best female he can conceivably get in a stable world. Presumably, she might be better than the female he gets in the stable pairing output by TMA.
• The Traditional Marriage Algorithm yields a matching at which each male gets his optimal female

Thm: TMA in a sequential way

Assume: At each time stamp, (every "tick" of the clock) there is exactly one event:
  • Event: a single man proposes, and if got rejected, his next proposal will be in next time stamp)

Note: The exact order is not crucial:
  • If both \( m_1, m_2 \) are proposing to \( f \), the result is the same independent of whom proposed first.

Thm: TMA produces a male-optimal pairing

Proof: Suppose, for a contradiction, that some male gets rejected by his optimal female during TMA.

Let \( t \) be the earliest time at which a male \( m_1 \) got rejected by his optimal female \( f \) (Florence)

Florence rejected \( m_1 \), because she said "maybe" a preferred male \( m_2 \).

\( m_2 \) had not yet been rejected by his optimal female (by the definition of \( t \)).

Therefore \( f \) is either the optimal female of \( m_2 \) Or \( f \) is higher the optimal female in his list.

That is, in any stable world, \( m_2 \) would either be married to \( f \) or to somebody lower on his list (definition of opt)

Let \( S \) be the matching at which \( (m_1, f) \) are married
  \( (S \) is NOT the result of the TMA)

Now consider \( (m_2, f) \) – they are a rouge couple. QED
The Pessimal male

• A female’s 
  pessimal male is the lowest
  ranked male for whom there is some
  stable matching which the female gets
  him.

• He is the worst male she can
  conceivably get in a stable world.

Thm: The TMA is female-pessimal.

Proof: We know it is male-optimal. \((m_1, f_1)\) is a couple in
TMA, \(\Rightarrow f_1 \) is \(m_1\) optimal female.
Suppose there is a stable pairing \(S\) where some female \(f_1\) does
worse than in \(TMA\).
– Let \(m_1\) be \(f_1\) husband in \(TMA\).
– Let \(m_2\) be \(f_1\) husband in \(S\)
  \((m_2, f_1)\) is a couple in \(S\) \(m_2\) is worse than \(m_1\)
– By assumption, \(m_1\) prefers \(f_1\) over his wife \(f_2\) in \(S\)
  • (since \(f_1\) is his optimal female)
– So \((m_1, f_1)\) is a rogue couple.
– Therefore, \(S\) is not stable. QED

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