This homework is due Thursday, April 9, at 3:30pm. Please upload a single PDF file containing your submission (ensuring scans of handwritten work are legible) on Gradescope by that time.

The questions are drawn from the material in class, and in Chapter 16 of the text, on greedy algorithms.

The homework is worth a total of 100 points. If point breakdowns are not given for each part of a problem, each part has equal weight.

For questions that ask you to design a greedy algorithm, prove that your algorithm is correct using a greedy augmentation lemma of the following form:

**Lemma** If a partial solution $P$ is contained in an optimal solution, then the greedy augmentation of $P$ is still contained in an optimal solution.

Prove the lemma using an exchange-style argument, where you transform the optimal solution so it contains the augmentation and then argue that the transformation does not worsen the solution value. Finally, prove correctness of your algorithm in a theorem that makes use of the lemma.

Remember to (a) start each problem on a new page, and (b) put your answers in the correct order. If you can’t solve a problem, state this, and write only what you know to be correct. Neatness and conciseness count.

(1) **(Counterexamples to greedy procedures)** (30 points) Prove that the following greedy procedures for the Activity Selection Problem are not correct. Each procedure considers the activities in a particular order, and selects an activity if it is compatible with those already chosen.

   (a) The activities are considered in order of increasing duration.
   (b) The activities are considered in order of increasing start-time.
   (c) The activities are considered in order of increasing number of overlaps with the remaining compatible activities. (This is a dynamically-determined order.)

   (Note: To prove an optimization procedure is not correct, it suffices to demonstrate a counterexample: namely, an instance of the problem that has a feasible solution that is better than the one the procedure outputs.)

(2) **(Trip refueling)** (30 points) Suppose you want to travel from city $A$ to city $B$ by car, following a fixed route. Your car can travel $m$ miles on a full tank of gas, and you have a map of the $n$ gas stations on the route between $A$ and $B$ that gives the number of miles between each station.

   Design a greedy algorithm to find a way to get from $A$ to $B$ without running out of gas that minimizes the total number of refueling stops, in $O(n)$ time. Prove that your algorithm finds an optimal sequence of stops.

(3) **(Minimizing average completion-time)** (40 points) Suppose you are given a collection of $n$ tasks that need to be scheduled. With each task, you are given its duration. Specifically, task $i$ takes $t_i$ units of time to execute, and can be started at any time. At any moment, only one task can be scheduled.

   The problem is to determine how to schedule the tasks so as to minimize their average completion-time. More precisely, if $c_i$ is the time at which task $i$ completes in a particular schedule, the average completion-time for the schedule is $\frac{1}{n} \sum_{1 \leq i \leq n} c_i$. 
(a) (40 points) Design a greedy algorithm that, given the task durations \( t_1, t_2, \ldots, t_n \), finds a schedule that minimizes the average completion-time. You may assume that once a task is started, it is run to completion. Your algorithm should take \( O(n \log n) \) time.

Analyze the running time of your algorithm, and prove that it is correct using a lemma of the required form.

(b) **(bonus) (20 points)** Suppose with each task we also have a release time \( r_i \), and that a task may not be started before its release time. Furthermore, tasks may be preempted, in that a scheduled task can be interrupted and later resumed, and this can happen repeatedly.

Design an algorithm that finds a schedule that minimizes the average completion-time in this new situation. Analyze its running time and prove that it is correct.

Note that Problem (3)(b) is a bonus question, and is not required.