CS 445

Dynamic Programming

Some of the slides are courtesy of Charles Leiserson with small changes by Carola Wenk

Dynamic Programming:
Example 1: Longest Common Subsequence

We look at sequences of characters (strings)

e.g. *x="ABCA"*

Def: A **subsequence** of *x* is an sequence obtained from *x* by possibly deleting some of its characters (but without changing their order

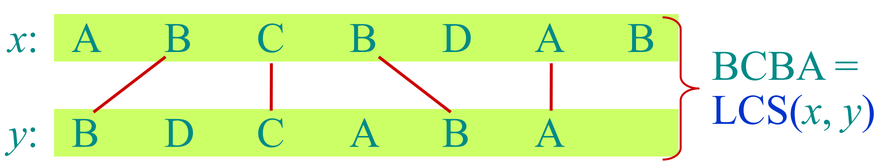
Examples:
"ABC", "ACA", "AA", "ABCA"

Def A **prefix** of *x*, denoted *x[1..m]*, is the sequence of the first *m* characters of *x*

Examples:
x[1..4]="ABCA" *x[1..3]="ABC"* *x[1..2]="AB"*
x[1..1]="A" *x[1..0]=""*

Longest Common Subsequence (LCS)

- Given two sequences $x[1\ .\ .\ m]$ and $y[1\ .\ .\ n]$, find a longest subsequence common to them both.



Different phrasing: Find a set of a maximum number of segments, such that
•Each segment connects a character of *x* to an identical character of *y*,
•Each character is used at most once
•Segments do not intersect.

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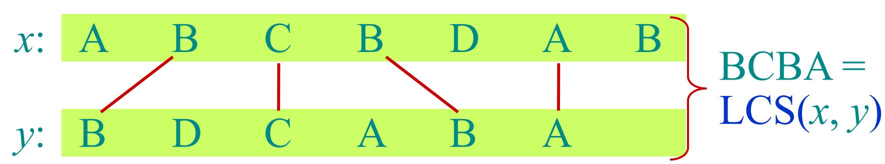

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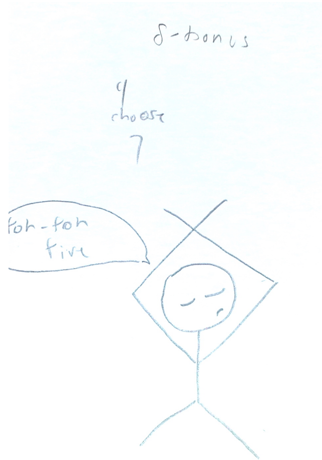
Cs445 salute



Brute-force LCS algorithm

Checking every subsequence of x whether it is also a subsequence of y.

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Analysis

- Checking $= \Theta(m+n)$ time per subsequence.
- 2^m subsequences of x

Worst-case running time $= \Theta((m+n)2^m)$
$=$ exponential time.

Towards a better algorithm

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2. Extend the algorithm to find the LCS itself.

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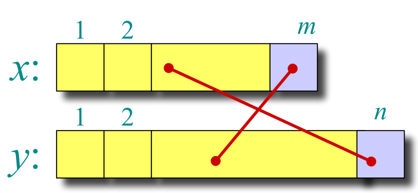
Strategy: Consider ***prefixes*** of x and y.

- Define $c[i, j] = |\text{LCS}(x[1 \,..\, i], y[1 \,..\, j])|$.
- Then, $c[m, n] = |\text{LCS}(x, y)|$.

Recursive formulation

Observation:
It is impossible that
x[m] is matched to an element in *y[1..n-1]* and simultaneously
y[n] is matched to an element in *x[1..m-1]*
(since it must create a pair of crossing segments).

Conclusion *– either x[m] is matched to y[n], or one at least of them is unmatched in* ***OPT.***
*{**OPT** – the optimal solution}*

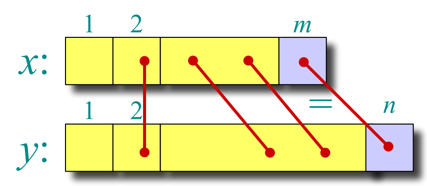


Recursive formaula

Lets just consider the last character of of x and of y

Case (I): $x[m] = y[n]$. Claim: $c[m, n]=c[m-1,n-1]+1$.

Proof.



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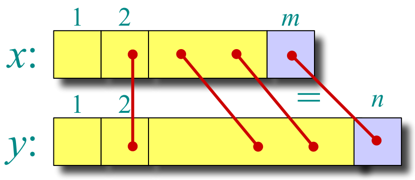


We claim that there is a max matching that matches $x[m]$ to $y[n]$.

Indeed, if $x[m]$ is matched to $y[k]$ (for $k<m$) then $y[n]$ is unmatched (otherwise we have two crossing segments). Hence we can obtain another matching of the same cardinality by matching $x[m]$ to $y[n]$.

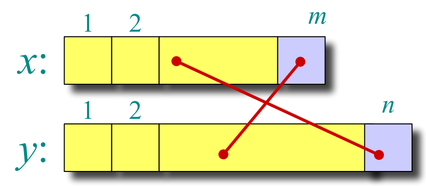
This implies that we can find an optimal matching of LCS($x[1..m-1]$ to y[$1..n-1$], and add the segment ($x[m],y[n]$).

So $c[m,n]=c[m-1,n-1]+1$

Recursive formulation-cont

Case (II): $x[m] \neq y[n]$ Claim: $c[m,n]=\max\{c[m,n-1], c[m-1,n]\}$

Recall - in LCS($x[1..m], y[1..n]$) it cannot be that **both** $x[m]$ and $y[n]$ are both matched.



If $x[m]$ is unmatched in OPT then

LCS($x[1..\mathbf{m}], y[1..\mathbf{n}]$)= LCS($x[1..\mathbf{m-1}], y[1..\mathbf{n}]$)

If $y[j]$ is unmatched in OPT then

LCS($x[1..\mathbf{m}], y[1..\mathbf{n}]$)= LCS($x[1..\mathbf{m}], y[1..\mathbf{n-1}]$)

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c[i,j] For general *i,j*

Since we only care for OPT matching the prefixes, then

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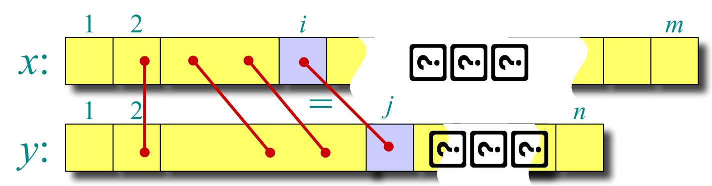

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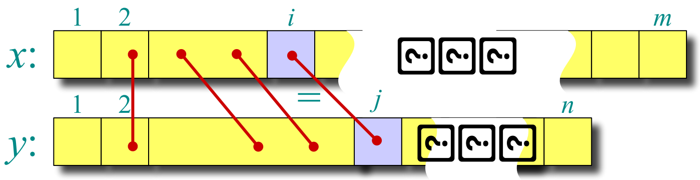

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This implies that we can match *x[1..i-1]* to y[*1..j-1*], and add the match (*x[i],y[j]*). So $c[i, j]=c[i-1,j-1]+1$

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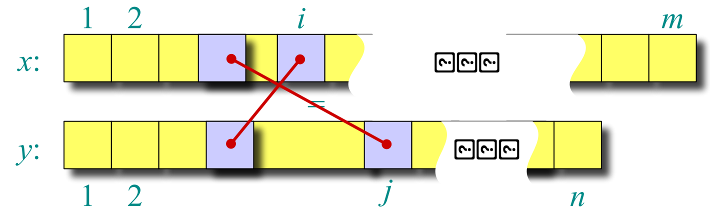

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If *y[j]* is unmatched then

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Dynamic-programming hallmark #1

Optimal substructure
An optimal solution to a problem (instance) contains optimal solutions to subproblems.

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If $z = \text{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.

Recursive algorithm for LCS

```
LCS(x, y, i, j)
   if ( i==0 or j=0) return 0
    if x[i] = y[ j]
          then return  LCS(x, y, i–1, j–1) + 1
          else return max{LCS(x, y, i–1, j), LCS(x, y, i, j–1)}
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To call the function LCS(x, y, m,n)

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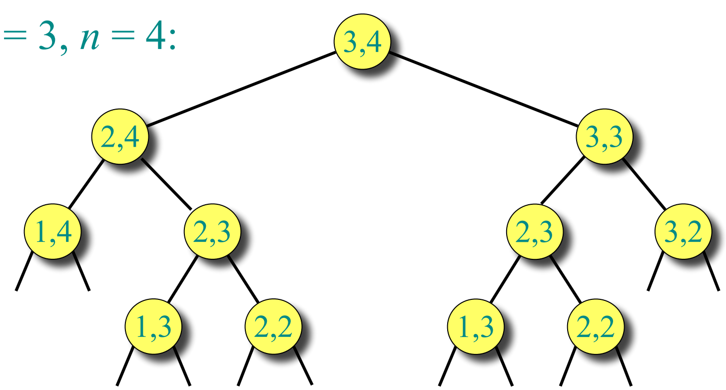
Worst-case: $x[i] \neq y[j]$, for all i,j in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

Recursion tree

$m = 3, n = 4$:



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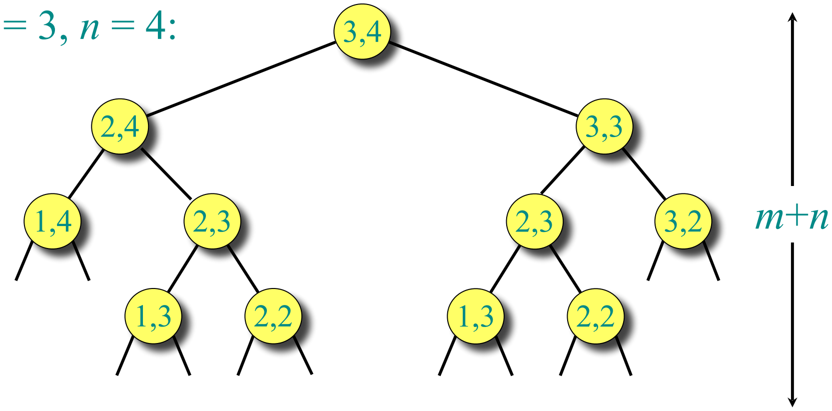

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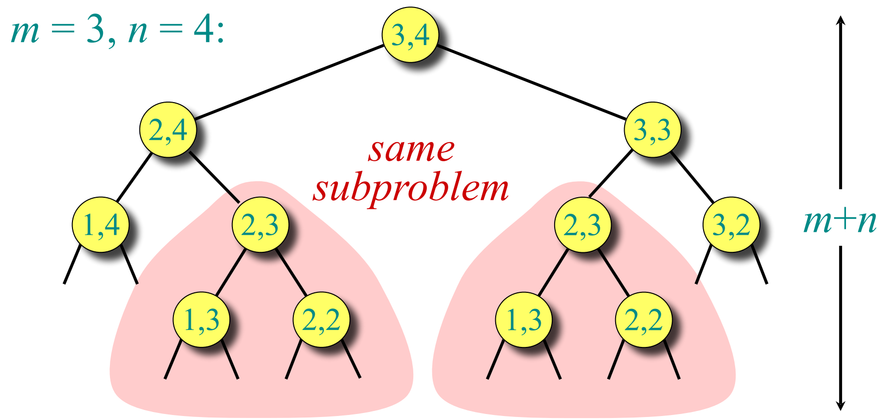

Height $= m + n \Rightarrow$ work potentially 2^{m+n} exponential.
but we're solving subproblems already solved!

Dynamic-programming hallmark #2

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A recursive solution contains a "small" number of distinct subproblems repeated many times.

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The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

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Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

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Time = $\Theta(mn)$ = constant work per table entry.
Space = $\Theta(mn)$.

LCS: Dynamic-programming algorithm

LCS(X,Y)="BCBA"

X=B D C A B A

Y=A B C B D A B

X= \ Y=		1 A	2 B	3 C	4 B	5 D	6 A	7 B
	0	0	0	0	0	0	0	0
1 B	0	0	1	1	1	1	1	1
2 D	0	0	1	1	1	2	2	2
3 C	0	0	1	2	2	2	2	2
4 A	0	1	1	2	2	2	3	3
5 B	0	1	2	2	3	3	3	4
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Reconstruction *z=LCS(x,y)*

IDEA: Compute the table bottom-up. Fill *z* backward.

Observation: *c[i;j]≥c[i-1;j]* and *c[i;j] ≥c[i;j-1]*
Proof Sketch: We use a longer prefix, so there are more chars to be match.

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		1 A	2 B	3 C	4 B	5 D	6 A	7 B
	0	0	0	0	0	0	0	0
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  if (*c[i;j]==c[i;j-1]*) *j--* ;
  else *i--* ;
}

		1 A	2 B	3 C	4 B	5 D	6 A	7 B
	0	0	0	0	0	0	0	0
1B	0	0	1	1	1	1	1	1
2D	0	0	1	1	1	2	2	2
3C	0	0	1	2	2	2	2	2
4A	0	1	1	2	2	2	3	3
5B	0	1	2	2	3	3	3	4
6A	0	1	2	2	3	3	4	4

Reconstruction *z=LCS(x,y)*

IDEA: Compute the table bottom-up. Fill *z* backward.

LCS(x,y)="BCBA"

x=B D C A B A

y=A B C B D A B

Observation: *c[i;j]≥c[i-1;j]* and *c[i;j] ≥c[i;j-1]*
Proof Sketch: We use a longer prefix, so there are more chars to be match.

LCS Reconstruction:
Set *i=m; j=n; k=c[i;j]*
While(*k>0*){
  if (*c[i;j]>c[i-1;j]* and *c[i;j]>c[i;j-1]*) {
    z[k] = x[i] ;
    i--; j-- ; k-- ;
  }else // *c[i;j]=c[i-1;j]* or *c[i;j]=c[i-1;j]*
  if (*c[i;j]==c[i;j-1]*) *j--* ;
  else *i--* ;
}

		1 A	2 B	3 C	4 B	5 D	6 A	7 B
	0	0	0	0	0	0	0	0
1B	0	0	1	1	1	1	1	1
2D	0	0	1	1	1	2	2	2
3C	0	0	1	2	2	2	2	2
4A	0	1	1	2	2	2	3	3
5B	0	1	2	2	3	3	3	4
6A	0	1	2	2	3	3	4	4

Reconstructing *z=LCS(X,Y)*

Another idea – While filling *c[]*, add arrows to each cell *c[i,j]* specifying which neighboring cell *c[i,j]* it got its value.
- *c[i,j].flag* = "\ " if *c[i,,j]=c[i-1;j-1]+1*
- *c[i,j].flag* = "↑ " if *c[i,,j]=c[i-1;j]*
- *c[i,j].flag* = "←" if *c[i,,j]=c[i-1;j]*

		A	B	C	B	D	A	B
	0	←0	0	0	0	0	0	0
B	0	↑0	1	←1	1	1	1	1
D	0	↑0	↑1	←1	←1	2	2	2
C	0	↑0	↑1	2	←2	←2	2	2
A	0	1	↑1	↑2	←2	2	3	3
B	0	↑1	2	←2	3	←3	3	4
A	0	1	↑2	←2	↑3	←3	4	←4

Example 2: Edit distance

Given strings *X,Y*, the **edit distance** ***ed***(*X,Y*) between *X* and *Y* is defined as the minimum number of operations that we need to perform on *X*, in order to obtain *Y*.

Defintion: An Operations (in this context) Insertion/Deletion/ Replacement of a **single** character.

Examples:
ed("aaba", "aaba") = 0
ed("aaa", "aaba") = 1
ed("aaaa", "abaa") = 1
ed("baaa", "") =4
ed("baaa", "aaab") =2

Note that the term "distance" is a bit misleading: We need both the **value** (how many operations) as well as knowing **which** operations.

Example 3':
``Priced" Edit distance *ed(X,Y)*

Assume also given
InsCost, - the cost of a single **insertion** into *x*.
DelCost - the cost of a single **deletion** from *x*, and
RepCost - the cost of **replacing** one character of *x* by a different character.

Definition: Given strings *X,Y*, the **edit distance** ***ed***(*X,Y*) between X and *Y* is the cheapest sequence of operations, starting on *X* and ending at *Y*.

Problem: Compute ***ed***(*X,Y*), (both the value and the optimal sequence of operations.)

Definition: ***c[i,j] = Cost(ed(X[1..i], Y[1..j])).***

Will first compute Cost(*c[m,n]*). Then will recover the sequence.

Thm:

Let *c[i,j]* = ed(*x[1..i], y[1..j]*).
Assume *c[i-1,j-1], c[i-1,j-1] , c[i-1,j]* are already computed.

If *X[i]=Y[j]* then *c[i,j] = c[i-1,j-1]*
Else // *X[i]≠Y[j]*
c[i,j] = min{
c[i-1,j-1]+*RepCost*, //convert *X[1..i-1]➔Y[1..j-1]*, and replace *y[j]* by *x[i]*
c[i-1, j] +*DelCost*, //delete *X[i]* and convert *X[1..i-1]➔ Y[1..j]*
c[i,j-1]+*InsCost* //convert *X[1..i,]➔ Y[1..j-1]*, and insert *Y[i]*
}
}

Algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

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```
ed(X, Y)
    for i=0 to m   c[i, 0] = i DelCost
    for j=0 to n   c[0, j] = j InsCost

    for i=1 to m
      for j=1 to n
        if (X[i] == Y[j] )
            then c[i, j] ← c[ i–1, j–1]
            else c[i, j] ←min{    c[ i-1 , j ]   +    DelCost,
                                  c[ i-1, j-1 ] +    RepCost,
                                  c[ i , j-1]    +    InsCost
                                  }
```

Algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

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                                  c[ i-1, j-1 ] +    RepCost,
                                  c[ i , j-1]    +    InsCost
                                  }
```
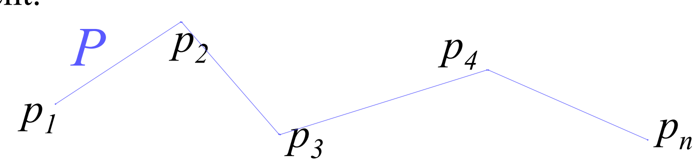

Time = $\Theta(m\ n)$ = constant work per table entry. Space = $\Theta(m\ n)$.
Homework: Compute the sequence of operations.
Compute which characters in x matches which chars in y.

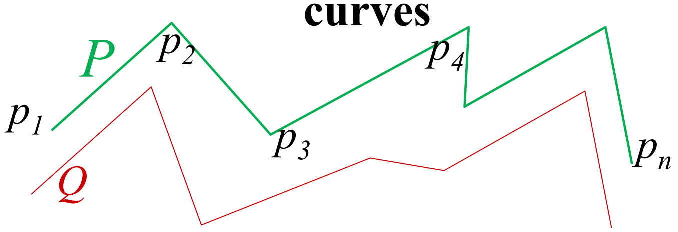
Polygonal Path - definition

We definite a polygonal path $P=\{p_1 \ldots p_n\}$ where

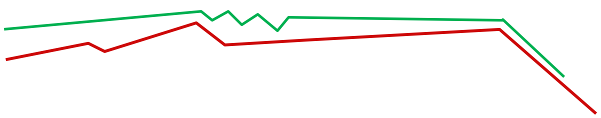
- Each vertex p_i is a point in the plane,
- Vertex p_1 is the first vertex , p_n is the last,
- Vertex p_i is connected to the next vertex p_{i+1} by a straight segment.



Good ways to measure distance between curves



- Should not be effected by how curves are sampled
- Should reflect the “order” of the points along the curves.

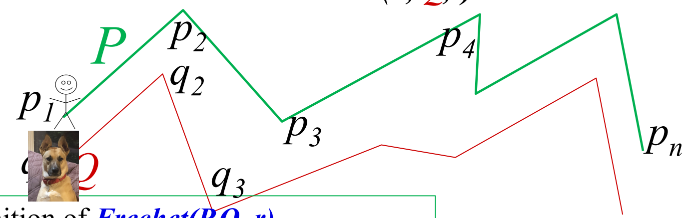


P[1..i] is the polygonal line with the first ***i*** vertices of *P*

Q*[1..j]* is the polygonal line with the first ***j*** vertices of *P*

Problem: Computing the Frechet Distance between polylines

Frechet(P, Q,r)



Definition of ***Frechet(P,Q, r)***

Assume a person walks on ***P={p_1…p_n}***

while a dog walks on ***Q={q_1..q_n }.***

r is the leash length (part of input).

The **person** starts at ***p_1*** and ends at ***p_n***

The **dog** starts at ***q_1*** and ends at ***q_n***

At each time stamp,

•either the **person** jumps to the next vertex

•Or the **dog** jumps to the next vertex

•Or **both** jumps to the next vertex

- Every instance they stop, we measure whether the distance between person↔dog (the length of the **leash)** ≤ ***r***.
- Frechet(P,Q,r)=YES if the answer is positive for all time stamps.
- (if not, a longer leash is need. If yes, maybe a shorter one is sufficient.
- So we could use binary search.

Problem: Computing the Frechet Distance between polylines

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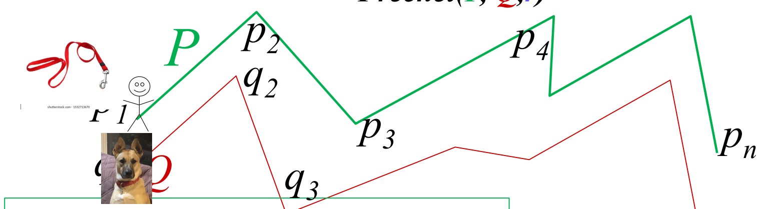

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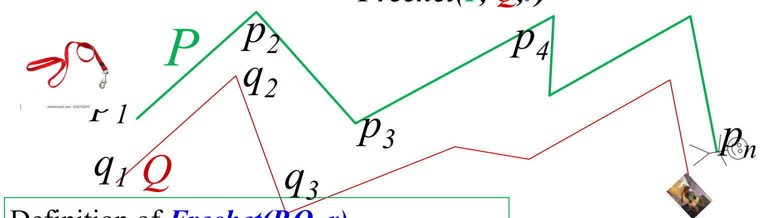

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- So we could use binary search.

Computing Frechet(P,Q,r)

Frechet(P,Q,r)
// c[1..n, 1..n] – boolean array
// ***c[i,j]=*** Frechet(P[1..*i*],Q[1..*j*], *r*)

Init:
c[1,1]= *(*$\|p_1 - q_1\| \le r$ *) (YES/NO)*
For *i=2* to *n* ***c[i,1]=*** *(*$\|p_i - q_1\| \le r$ *) AND c[i-1,1] (YES/NO)*
For *j=2* to *n* ***c[1,j]=*** *(*$\|p_1 - q_j\| \le r$ *) AND c[1,j-1]*

Computing Frechet (P,Q,r) (cont.)

// c[1..n, 1..n] – boolean array

Init- previous slide

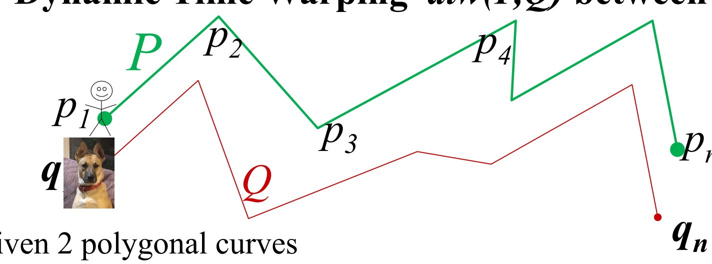
For *i=2* to *n*
For *j=2* to *n*
c[i,j] *=* ($\|p_i - q_j\| \le r$) **AND**
{ ***c[i-1,j-1], // both jumps***
OR ***c[i-1, j]*** , // person jumped from p_{i-1} to p_i , dog stays at q_j
OR ***c[i,j-1] .*** // person stayed at p_i , dog jumped from q_{j-1} *to* q_{j-}
}
Return *c[n.n]*
Note – this is only the cost (that is the distance itself. We still need to find what is the series of steps that yield this cost

Comments

- This is actually the **Discrete** Frechet Distance (only distances between vertices counts). We do not discuss the **continuous** version.
- This is only the Decision problem – we actually want the shortest leash. We could use a binary search to approximate it. Exact algorithm outside the scope of this course
- If person/dog could move backward, the problem is called the **weak** Frechet.

Maurice René Fréchet

Problem: Computing Dynamic Time Warping *dtw(P,Q)* between polylines

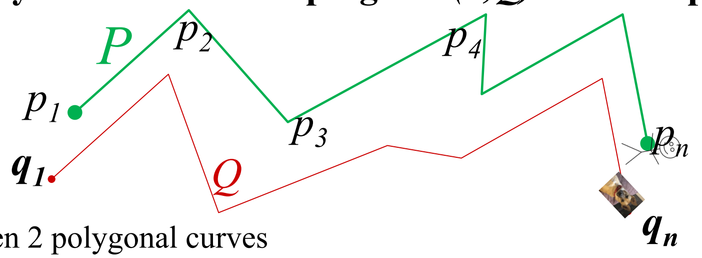


Given 2 polygonal curves
P={p₁…pₙ} and ***Q={q₁..qₘ},***
The input is the locations of their vertices (e.g. GIS coordinates)

How similar are *P* to *Q* ?

Need to come up with a number ***dtw(P,Q)***?
So if *dtw(P,Q)< dtw(P,Q')*, then ***P*** is more similar to ***Q***

Problem: Computing Dynamic Time Warping *dtw(P,Q)* between polylines

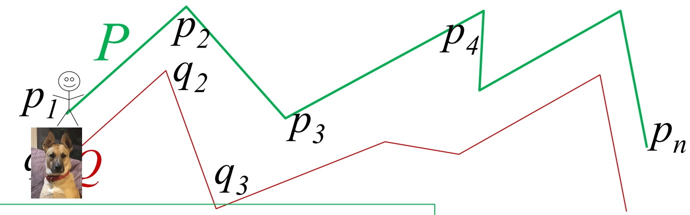


Given 2 polygonal curves
$P=\{p_1\ldots p_n\}$ and $Q=\{q_1..q_m\}$,
The input is the locations of their vertices (e.g. GIS coordinates)

How similar are *P* to *Q* ?

Need to come up with a number ***dtw(P,Q)***?
So if *dtw(P,Q)< dtw(P,Q')*, then ***P*** is more similar to ***Q***

Dynamic Time Warping *dtw(P,Q)*



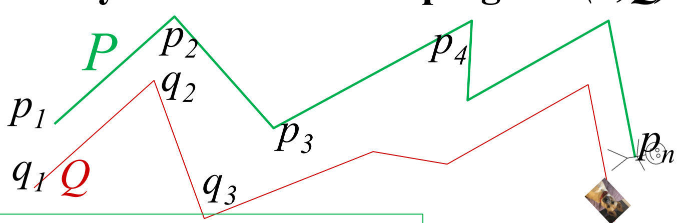
Definition of ***dtw(P,Q)***
Assume a person walks on $P=\{p_1\ldots p_n\}$
while a dog walks on $Q=\{q_1..q_m\}$.

They **person** starts at p_1 and ends at p_n
They **dog** starts at q_1 and ends at q_n

At each time stamp,
•either the **person** jumps to the next vertex
•Or the **dog** jumps to the next vertex
•Or **both** jumps to the next vertex

- Every instance they stop, we measure the distance (the length of the **leash**) person↔dog.
- We sum the lengths of all leashes.
- dtw(P,Q) is the smallest sum (over all possible sequences)

Dynamic Time Warping *dtw(P,Q)*



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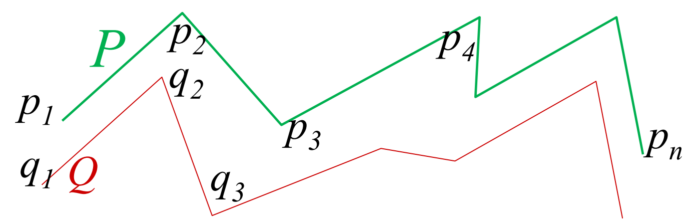
- We sum the lengths of all leashes.
- dtw(P,Q) is the smallest sum (over all possible sequences)

Motivation:



Definition of ***dtw(P,Q)***
Assume a person walks on $P=\{p_1\ldots p_n\}$
while a dog walks on $Q=\{q_1..q_m\}$.

Distance between trajectoris enables finding nearest neighbor, and clustering

But two very similar trajectories might have vertices in very different places

DTW is used in
- Signal processing (speech reco)
- Signature verification
- Analysis of vehicles trajectories for roads networks
- **Improving locations-based services**
- **Animals migrations patters**
- Stocks analysis

Thm 1:

Let $c[i,j]$ = dtw($P[1..i]$, $Q[1..j]$).

Let $\| p_i - q_j \|$ be the between the points p_i *and* q_j

That is, the length of the leash.

For every $i>1$,$j>1$

$c[1,1] = \| p_1 - q_1 \|$

$c[1,j] = c[1,j-1] + \| p_1 - q_j \|$

$c[i,1] = c[i-1,1] + \| p_i - q_1 \|$

Thm 2:

Assume at some time, the person is at p_i while dog at q_j.

Assume $i>1$ and $j>1$.

What (might have) happened one step ago ?

Three possibilities

Both person and the dog jumped (from p_{i-1} *and from* q_j *) OR*

Person jumped from p_{i-1} to p_i , dog stays at q_j *OR*

Person stayed at p_i , dog jumped from q_{j-1} *to* q_{j-}

Thm 2 cont:

Let $c[i,j]$ = dtw($P[1..i]$, $Q[1..j]$).

If $i>1$ and $j>1$ then

$c[i,j] = \| p_i - q_j \| +$
min{
c[i-1,j-1], // both jumps
c[i-1, j] , // person jumped from p_{i-1} to p_i , dog stays at q_j
c[i,j-1] . // person stayed at p_i , dog jumped from q_{j-1} *to* q_{j-}
}

Since we are not sure that when the person is at p_i the dog is at q_j we will compute all such pairs i,j – one of them must happened

Algorithm for computing dtw(P,Q)

Init according to Thm 1.

```
Fot i=2 to n
   For j=2 to n
      c[i,j] = || p_i - q_j || +
         min{
         c[i-1,j-1], // both jumps
         c[i-1, j ] , // person jumped from p_{i-1} to p_i , dog stays at q_j
         c[ i,j-1] // person stayed at p_i , dog jumped from q_{j-1} to q_{j-}
         }
Return c[n.n]
```

Note – this is only the cost (that is the distance itself. We still need to find what is the series of steps that yield this cost

Dynamic-programming hallmark #1

(we saw this slide already)

Optimal substructure
An optimal solution to a problem (instance) contains optimal solutions to subproblems.

Dynamic-programming hallmark #1

(we saw this slide already)

Optimal substructure
An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If z = LCS(x, y), then any prefix of z is an LCS of a prefix of x and a prefix of y.

Dynamic-programming hallmark #2

Overlapping subproblems
A recursive solution contains a "small" number of distinct subproblems repeated many times.

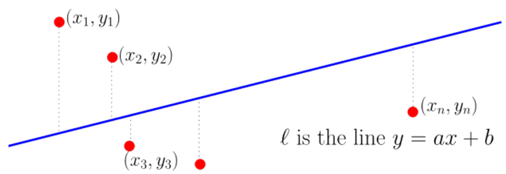
Dynamic-programming hallmark #2

Overlapping subproblems
A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Another application of DP: **Clustering**

(source: Kleinberg & Tardos 6.3)

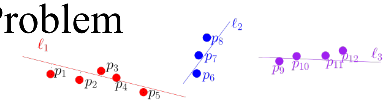


- Given points $P = \{(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)\}$ find a line minimizing $Err(\ell, P)$
- $$Err(\ell, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$ that is, the sum of squares of vertical distances from each (x_i, y_i) to ℓ.
- Solution

$$a = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2}$$

$$b = \frac{\sum y_i - a \sum x_i}{n}$$

Clustering Problem



- Given points $P = (p_1, p_2, \dots p_n)$ sorted from left to right, and a panelty R, find optimal k, and partition of P into k **runs**

$(p_1, p_2 \dots p_{i_1})(p_{i_1+1}, p_{i_1+2} \dots p_{i_2}), (p_{i_2+1}, \dots p_{i_3}) \dots (p_{i_{k-1}+1} \dots p_n)$

and lines $\ell_1 \dots \ell_k$ (one per each run) So that the sum

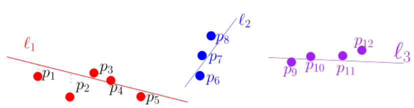
$$R + Err(\ell_1, \{p_1, p_2 \dots p_{i_1}\}) +$$
$$R + Err(\ell_2, \{p_{i_1+2} \dots p_{i_2}\}) +$$
$$\vdots$$
$$R + Err(\ell_k, \{p_{i_{k-1}+1} \dots p_n\})$$

is as small as possible

The penalty R

- Note that if R=0, we will probably use n/2 runs $(p_1\ p_2), (p_3, p_4), \dots (p_{n-1}, p_n)$.
- If R is huge, we can afford only one penalty, so only one run $(p_1 \dots p_n)$.
- In the example, $k=3$, $i_1=5$, $i_2=8$
- Worth mentioning: There is no correct value of the penalty R. Instead, think that the user could slowly increase R from 0 to ∞, watch the number of clusters increases, and stop when the lines seems appropriate.
- The Geogabra applet link could help visualizing this process

Algorithm



- Preprocessing: for every pair of i and j (where $j < i$) compute the line $\ell_{j,i}$ that best fit the points $\{p_j, p_{j+1}, p_{j+2} \cdots p_i\}$
- Let c[i] = cost of the cost of the opt clustering of the points $\{p_1 \dots p_i\}$. This term includes both the sum of errors and the sum of penalties. At the i'th step of the algorithm, we assume that $c[0], c[1], c[2], \dots c[i-1]$ are already computed, and using these values, we will compute ***c[i].***
- Init: c[0]=0
- for i=2 to n do {
  - $c[i] = \min\{c[j] + R + e[j+1, i]$ such that $j = 0,1,2 \dots i-1\}$
  - $c[i]$ could also `remember' for which value of j the minimum is obtained.

}

Idea: p_i must belong to a cluster. We pay R for this cluster. The inner loop finds what is the best point p_{j+1} to be the leftmost point of this cluster.

Summarizing

- The algorithm takes *O(n³)* and *O(n²)* space
- (for preprocessing *d[j,i]*)
- Note – we did not discuss how to reconstruct the solution itself. We only calculated its cost