CS 445

Dynamic Programming

Some of the slides are courtesy of Charles Leiserson with small changes by Carola Wenk

---

Dynamic Programming:
Example 1: Longest Common Subsequence

We look at sequences of characters (strings)

e.g. *x="ABCA"*

Def: A **subsequence** of *x* is an sequence obtained from *x* by possibly deleting some of its characters (but without changing their order

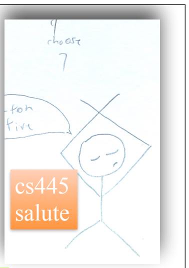
Examples:
"ABC", "ACA", "AA", "ABCA"

Def A **prefix** of *x*, denoted *x[1..m]*, is the sequence of the first *m* characters of *x*

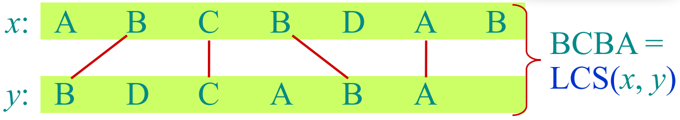
Examples:
x[1..4]="ABCA" x[1..3]="ABC" x[1..2]="AB"
x[1..1]="A" x[1..0]=""

Longest Common Subsequence (LCS) problem:

- Given two sequences $x[1 \,.\,.\, m]$ and $y[1 \,.\,.\, n]$, find a longest subsequence common to them both.

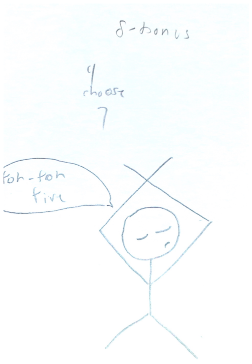


"a" *not* "the"



Different phrasing: Find a set of a maximum number of segments, such that
•Each segment connects a character of *x* to an identical character of *y*,
•Each character is used at most once
•Segments do not intersect.

Cs445 salute



Brute-force LCS algorithm

Checking every subsequence of x whether it is also a subsequence of y.

Analysis

- Checking $= \Theta(m+n)$ time per subsequence.
- 2^m subsequences of x

Worst-case running time $= \Theta((m+n)2^m)$
$=$ exponential time.

Towards a better algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

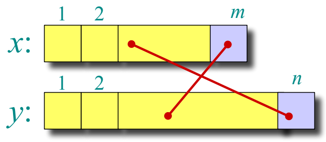
Strategy: Consider ***prefixes*** of x and y.

- Define $c[i, j] = |\text{LCS}(x[1 \,.\,.\, i], y[1 \,.\,.\, j])|$.
- Then, $c[m, n] = |\text{LCS}(x, y)|$.

Recursive formulation

Observation:
It is impossible that
x[m] is matched to an element in *y[1..n-1]* and
simultaneously
y[n] is matched to an element in *x[1..m-1]*
(since it must create a pair of crossing segments).

Conclusion *– either x[m] is matched to y[n], or one at least of them is unmatched in* ***OPT.***
*{****OPT*** *– the optimal solution}*

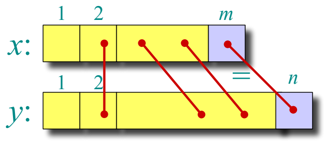


Recursive formaula

Lets just consider the last character of of x and of y

Case (I): $x[m] = y[n]$. Claim: $c[m, n]=c[m-1,n-1]+1$.

Proof.



We claim that there is a max matching that matches *x[m]* to *y[n].*

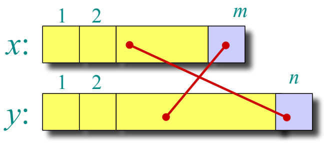
Indeed, if *x[m]* is matched to *y[k]* (for $k<m$) then *y[n]* is unmatched (otherwise we have two crossing segments). Hence we can obtain another matching of the same cardinality by matching *x[m]* to *y[n]*.

This implies that we can find an optimal matching of
LCS(*x[1..m-1]* to y[*1..n-1*], and add the segment (*x[m],y[n]*).
So $c[m,n]=c[m-1,n-1]+1$

Recursive formulation-cont

Case (II): $x[m] \neq y[n]$ Claim: $c[m,n]=\max\{c[m,n-1], c[m-1,n]\}$

Recall - in LCS($x[1 \,.\,.\, m], y[1 \,.\,.\, n]$) it cannot be that **both** $x[m]$ and $y[n]$ are both matched.

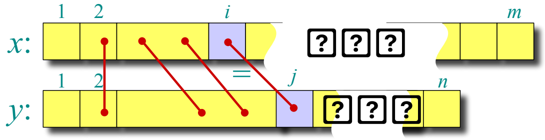


If *x[m]* is unmatched in OPT then
LCS($x[1 \,.\,.\, \mathbf{m}], y[1 \,.\,.\, \mathbf{n}]$)= LCS($x[1 \,.\,.\, \mathbf{m-1}], y[1.\,.\mathbf{n}]$)
If *y[j]* is unmatched in OPT then
LCS($x[1 \,.\,.\, \mathbf{m}], y[1 \,.\,.\, \mathbf{n}]$)= LCS($x[1 \,.\,.\, \mathbf{m}], y[1 \,.\,.\, \mathbf{n-1}]$)

So $c[m,n]= \max\{c[m–1, n], c[m, n–1]\}$

c[i,j] For general *i,j*

Since we only care for OPT matching the prefixes, then
Case (I): $x[i] = y[j]$.
Claim: if $x[i] = y[j]$ then $c[i, j]=c[i-1,j-1]+1$.



We claim that there is a max matching that matches $x[i]$ to $y[j]$.

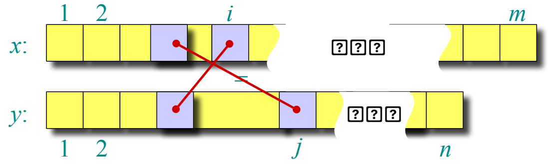
Indeed, if $x[i]$ is matched to $y[k]$ (for $k<j$) then $y[j]$ is unmatched (otherwise we have two crossing segments). Hence we can obtain another matching of the same cardinality by match $x[i]$ to $y[j]$.

This implies that we can match $x[1..i-1]$ to $y[1..j-1]$, and add the match $(x[i],y[j])$. So $c[i, j]=c[i-1,j-1]+1$

Recursive formulation-cont

Case (II): if $x[i] \neq y[j]$ then $c[i, j]=\max\{c[i-1, j], c[i, j-1]\}$

Recall - in LCS($x[1 . . i], y[1 . . j]$) it cannot be that **both** $x[i]$ and $y[j]$ are both matched.



If $x[i]$ is unmatched then
$$\text{LCS}(x[1 . . \boldsymbol{i}], y[1 . . \boldsymbol{j}])= \text{LCS}(x[1 . . \boldsymbol{i-1}], y[1 . . \boldsymbol{j}])$$
If $y[j]$ is unmatched then
$$\text{LCS}(x[1 . . \boldsymbol{i}], y[1 . . \boldsymbol{j}])= \text{LCS}(x[1 . . \boldsymbol{i}], y[1 . . \boldsymbol{j-1}])$$

So $c[i, j]= \max\{c[i-1, j], c[i, j-1]\}$

Dynamic-programming hallmark #1

Optimal substructure
An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If z = LCS(x, y), then any prefix of z is an LCS of a prefix of x and a prefix of y.

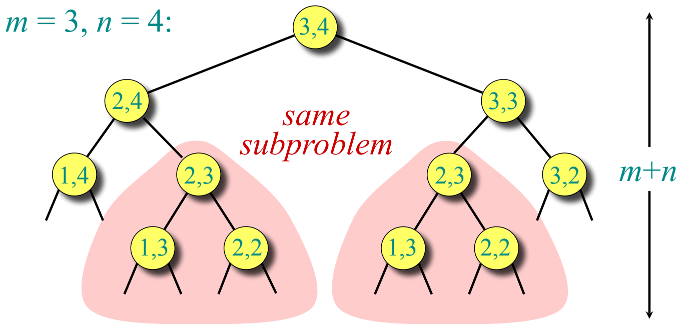
Recursive algorithm for LCS

```
LCS(x, y, i, j)
   if ( i==0 or j=0) return 0
     if x[i] = y[ j]
          then return  LCS(x, y, i–1, j–1) + 1
          else return max{LCS(x, y, i–1, j), LCS(x, y, i, j–1)}
```

To call the function LCS(x, y, m,n)

Worst-case: $x[i] \neq y[j]$, for all i,j in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

Recursion tree



Height = $m + n$ ⇒ work potentially 2^{m+n} exponential.
but we're solving subproblems already solved!

Dynamic-programming hallmark #2

> ***Overlapping subproblems***
> *A recursive solution contains a "small" number of distinct subproblems repeated many times.*

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

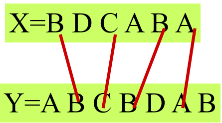
```
LCS(x, y)
    for i=0 to m   c[i, 0] = 0
    for j=0 to n   c[0, j] = 0

    for i=1 to m
      for j=1 to n
        if (x[i] = y[j] )
          then c[i, j] ← c[ i–1, j–1]  + 1
          else c[i, j] ← max{ c[ i–1, j],  c[i, j–1] }
```

Time = $\Theta(mn)$ = constant work per table entry.

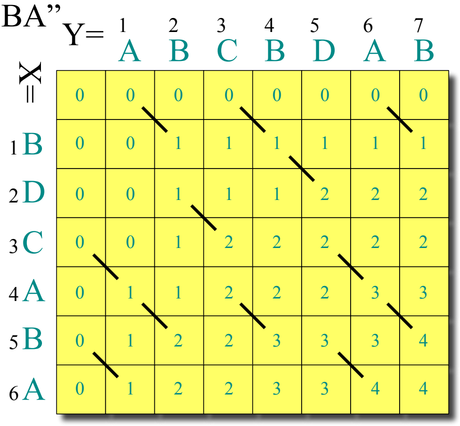
Space = $\Theta(mn)$.

LCS: Dynamic-programming algorithm

LCS(X,Y)=“BCBA”


X=B D C A B A

Y=A B C B D A B



X= \ Y=		1 A	2 B	3 C	4 B	5 D	6 A	7 B
	0	0	0	0	0	0	0	0
1 B	0	0	1	1	1	1	1	1
2 D	0	0	1	1	1	2	2	2
3 C	0	0	1	2	2	2	2	2
4 A	0	1	1	2	2	2	3	3
5 B	0	1	2	2	3	3	3	4
6 A	0	1	2	2	3	3	4	4

Reconstruction *z=LCS(x,y)*

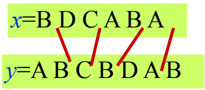
IDEA: Compute the table bottom-up. Fill *z* backward.

Observation: $c[i;j] \geq c[i-1;j]$ and $c[i;j] \geq c[i;j-1]$

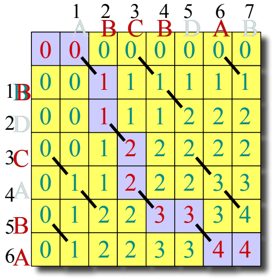
Proof Sketch: We use a longer prefix, so there are more chars to be match.

```
LCS Reconstruction:
Set i=m;  j=n;  k=c[i;j]
While(k>0){
  if (c[i;j]>c[i-1;j] and c[i;j]>c[i;j-1] ) {
        z[k] = x[i] ;
        i--; j-- ; k--  ;
  }else // c[i;j]=c[i-1;j] or c[i;j]=c[i-1;j]
  if  (c[i;j]==c[i;j-1])  j-- ;
  else i--   ;
}
```

LCS(x,y)=“BCBA”


x=B D C A B A

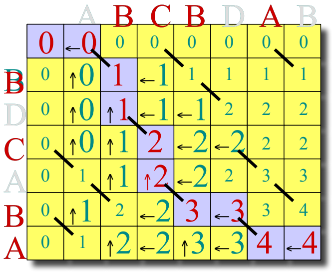
y=A B C B D A B



		1 A	2 B	3 C	4 B	5 D	6 A	7 B
	0	0	0	0	0	0	0	0
1B	0	0	1	1	1	1	1	1
2D	0	0	1	1	1	2	2	2
3C	0	0	1	2	2	2	2	2
4A	0	1	1	2	2	2	3	3
5B	0	1	2	2	3	3	3	4
6A	0	1	2	2	3	3	4	4

Reconstructing *z=LCS(X,Y)*

Another idea – While filling *c[],* add arrows to each cell *c[i,j]* specifying which neighboring cell *c[i,j]* it got its value.
- *c[i,j].flag* = “\ “ if *c[i,,j]=c[i-1;j-1]+1*
- *c[i,j].flag* = “↑ “ if *c[i,,j]=c[i-1;j]*
- *c[i,j].flag* = “←“ if *c[i,,j]=c[i-1;j]*



		A	B	C	B	D	A	B
	0	←0	0	0	0	0	0	0
B	0	↑0	1	←1	1	1	1	1
D	0	↑0	↑1	←1	←1	2	2	2
C	0	↑0	↑1	2	←2	←2	2	2
A	0	1	↑1	↑2	←2	2	3	3
B	0	↑1	2	←2	3	←3	3	4
A	0	1	↑2	←2	↑3	←3	4	←4

Example 2: Edit distance

Given strings *X,Y*, the **edit distance** ***ed***(*X,Y*) between *X* and *Y* is defined as the minimum number of operations that we need to perform on *X*, in order to obtain *Y*.

Defintion: An Operations (in this context) Insertion/Deletion/ Replacement of a **single** character.

Examples:

ed(“aaba”, “aaba”) = 0

ed(“aaa”, “aaba”) = 1

ed(“aaaa”, “abaa”) = 1

ed(“baaa”, “”) =4

ed(“baaa”, “aaab”) =2

Note that the term “distance” is a bit misleading: We need both the **value** (how many operations) as well as knowing **which** operations.

Example 3’:
``Priced” Edit distance *ed(X,Y)*

Assume also given
- *InsCost,* - the cost of a single **insertion** into *x.*
- *DelCost* - the cost of a single **deletion** from *x,* and
- *RepCost* - the cost of **replacing** one character of *x* by a different character.

Definition: Given strings *X,Y*, the **edit distance** ***ed***(*X,Y*) between X and *Y* is the cheapest sequence of operations, starting on *X* and ending at *Y.*

Problem: Compute ***ed***(*X,Y*), (both the value and the optimal sequence of operations.)

Definition: ***c[i,j] = Cost(ed(X[1..i], Y[1..j])).***

Will first compute Cost(*c[m,n]*). Then will recover the sequence.

Thm:

Let $c[i,j] = ed(x[1..i], y[1..j])$.

Assume $c[i-1,j-1], c[i-1,j-1] , c[i-1,j]$ are already computed.

If $X[i]=Y[j]$ then $c[i,j] = c[i-1,j-1]$

Else // $X[i]\neq Y[j]$

$c[i,j] = \min\{$

$c[i-1,j-1]+RepCost$, //convert $X[1..i-1] \rightarrow Y[1..j-1]$, and replace $y[j]$ by $x[i]$

$c[i-1, j] +DelCost$, //delete $X[i]$ and convert $X[1..i-1] \rightarrow Y[1..j]$

$c[i,j-1]+InsCost$ //convert $X[1..i,] \rightarrow Y[1..j-1]$, and insert $Y[i]$

$\}$

$\}$

Algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```
ed(X, Y)
    for i=0 to m   c[i, 0] = i DelCost
    for j=0 to n   c[0, j] = j InsCost

    for i=1 to m
      for j=1 to n
        if (X[i] == Y[j] )
          then c[i, j] ← c[ i–1, j–1]
          else c[i, j] ←min{   c[ i-1 , j ]   +   DelCost,
                               c[ i-1, j-1 ] +   RepCost,
                               c[ i , j-1]   +   InsCost
                               }
```

Time = $\Theta(m\ n)$ = constant work per table entry. Space = $\Theta(m\ n)$.

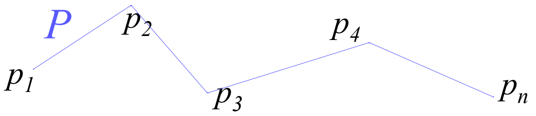
Homework: Compute the sequence of operations.

Compute which characters in x matches which chars in y.

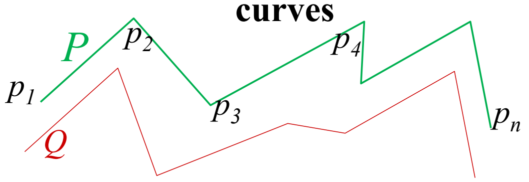
Polygonal Path - definition

We definite a polygonal path $P=\{p_1\ldots p_n\}$ where

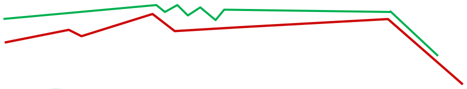
- Each vertex p_i is a point in the plane,
- Vertex p_1 is the first vertex , p_n is the last,
- Vertex p_i is connected to the next vertex p_{i+1} by a straight segment.



Good ways to measure distance between curves



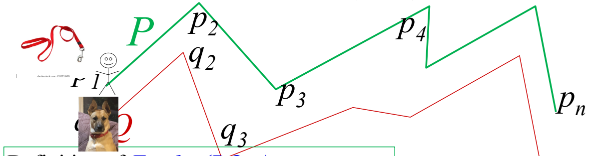
- Should not be effected by how curves are sampled
- Should reflect the "order" of the points along the curves.



$P[1..i]$ is the polygonal line with the first ***i*** vertices of *P*

$Q[1..j]$ is the polygonal line with the first ***j*** vertices of *P*

Problem: Computing the Frechet Distance between polylines *Frechet(P, Q,r)*



Definition of ***Frechet(P,Q, r)***
Assume a person walks on $P=\{p_1\ldots p_n\}$
while a dog walks on $Q=\{q_1..q_n\}$.
r is the leash length (part of input).
The **person** starts at p_1 and ends at p_n
The **dog** starts at q_1 and ends at q_n

At each time stamp,
•either the **person** jumps to the next vertex
•Or the **dog** jumps to the next vertex
•Or **both** jumps to the next vertex

- Every instance they stop, we measure whether the distance between person↔dog (the length of the **leash)** $\leq r$.
- Frechet**(P,Q,r)=YES** if the answer is positive for all time stamps.
- (if not, a longer leash is need. If yes, maybe a shorter one is sufficient.
- So we could use binary search.

Computing Frechet(P,Q,r)

```
Frechet(P,Q,r)
// c[1..n, 1..n] –  boolean array
// c[i,j]= Frechet(P[1..i],Q[1..j ],  r  )

Init:
c[1,1]=  ($\| p_1 - q_1 \| \leq r$ ) (YES/NO)
For i=2 to n  c[i,1]=  ($\| p_i - q_1 \| \leq r$ ) AND c[i-1,1]  (YES/NO)
For j=2 to n  c[1,j]=  ( $\| p_1 - q_j \| \leq r$ ) AND c[1,j-1]
```

Computing Frechet (P,Q,r) (cont.)

```
// c[1..n, 1..n] –  boolean array

Init- previous slide

For i=2 to n
    For j=2 to n
        c[i,j] = ( || $p_i$ - $q_j$ || ≤ r) AND
        {    c[i-1,j-1], // both jumps
         OR          c[i-1, j ] , // person jumped from $p_{i-1}$ to $p_i$ , dog stays at $q_j$
         OR          c[ i,j-1] . // person stayed at $p_i$ , dog jumped from $q_{j-1}$ to $q_{j-}$.
           }
Return c[n.n]
```

Note – this is only the cost (that is the distance itself. We still need to find what is the series of steps that yield this cost

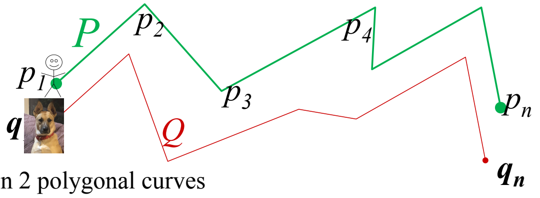
---

Comments

- This is actually the **Discrete** Frechet Distance (only distances between vertices counts). We do not discuss the **continuous** version.
- This is only the Decision problem – we actually want the shortest leash. We could use a binary search to approximate it. Exact algorithm outside the scope of this course
- If person/dog could move backward, the problem is called the **weak** Frechet.

Maurice René Fréchet

Problem: Computing Dynamic Time Warping *dtw(P,Q)* between polylines



Given 2 polygonal curves

$P=\{p_1…p_n\}$ and *$Q=\{q_1..q_m\}$*,

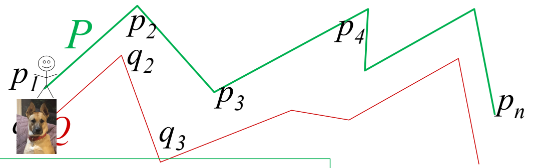
The input is the locations of their vertices (e.g. GIS coordinates)

How similar are *P* to *Q* ?

Need to come up with a number ***dtw(P,Q)***?

So if *dtw(P,Q)< dtw(P,Q')*, then ***P*** is more similar to ***Q***

Dynamic Time Warping *dtw(P,Q)*



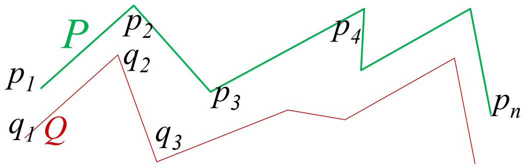
Definition of ***dtw(P,Q)***
Assume a person walks on $P=\{p_1\ldots p_n\}$
while a dog walks on $Q=\{q_1..q_m\}$.

They **person** starts at p_1 and ends at p_n
They **dog** starts at q_1 and ends at q_n

At each time stamp,
•either the **person** jumps to the next vertex
•Or the **dog** jumps to the next vertex
•Or **both** jumps to the next vertex

- Every instance they stop, we measure the distance (the length of the **leash**) person↔dog.
- We sum the lengths of all leashes.
- **dtw(P,Q)** is the smallest sum (over all possible sequences)

Motivation:



Definition of ***dtw(P,Q)***
Assume a person walks on $P=\{p_1\ldots p_n\}$
while a dog walks on $Q=\{q_1..q_m\}$.

Distance between trajectoris enables finding nearest neighbor, and clustering

But two very similar trajectories might have vertices in very different places

DTW is used in
- Signal processing (speech reco)
- Signature verification
- Analysis of vehicles trajectories for roads networks
- **Improving locations-based services**
- ***Animals migrations patters***
- Stocks analysis

Thm 1:

Let $c[i,j]$ = dtw($P[1..i]$, $Q[1..j]$).

Let $\| p_i - q_j \|$ be the between the points p_i *and* q_j
That is, the length of the leash.

For every $i>1$,$j>1$
$c[1,1]= \| p_1 - q_1 \|$

$c[1,j] = c[1,j-1] + \| p_1 - q_j \|$

$c[i,1] = c[i-1,1] + \| p_i - q_1 \|$

Thm 2:

Assume at some time, the person is at p_i while dog at $q_{j.}$
Assume $i>1$ and $j>1$.

What (might have) happened one step ago ?

Three possibilities

Both person and the dog jumped (from p_{i-1} *and from* q_j *) OR*
Person jumped from p_{i-1} to p_i , dog stays at q_j *OR*
Person stayed at p_i , dog jumped from q_{j-1} *to* q_{j-}

Thm 2 cont:

Let *c[i,j]* = dtw(*P[1..i], Q[1..j]*).

If $i>1$ and $j>1$ then

```
c[i,j] = || $p_i$ - $q_j$ || +
        min{
        c[i-1,j-1], // both jumps
        c[i-1, j ] , // person jumped from $p_{i-1}$ to $p_i$ , dog stays at $q_j$
        c[ i,j-1] . // person stayed at $p_i$ , dog jumped from $q_{j-1}$ to $q_{j-}$
        }
```

Since we are not sure that when the person is at p_i the dog is at q_j we will compute all such pairs *i,j* – one of them must happened

---

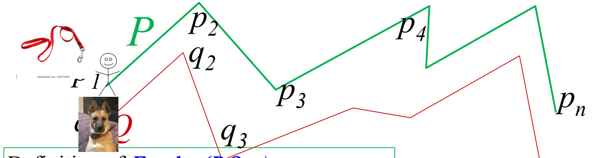
Algorithm for computing dtw(P,Q)

Init according to Thm 1.

```
Fot i=2 to n
    For j=2 to n
        c[i,j] = || $p_i$ - $q_j$ || +
            min{
            c[i-1,j-1], // both jumps
            c[i-1, j ] , // person jumped from $p_{i-1}$ to $p_i$ , dog stays at $q_j$
            c[ i,j-1] // person stayed at $p_i$ , dog jumped from $q_{j-1}$ to $q_{j-}$
            }
Return c[n.n]
```

Note – this is only the cost (that is the distance itself. We still need to find what is the series of steps that yield this cost

Problem: Computing the Frechet Distance between polylines
Frechet(P, Q,r)



Definition of ***Frechet(P,Q, r)***
Assume a person walks on $P=\{p_1\ldots p_n\}$
while a dog walks on $Q=\{q_1..q_n\}$.
r is the leash length (part of input).
The **person** starts at p_1 and ends at p_n
The **dog** starts at q_1 and ends at q_n

At each time stamp,
•either the **person** jumps to the next vertex
•Or the **dog** jumps to the next vertex
•Or **both** jumps to the next vertex

- Every instance they stop, we measure whether the distance between person↔dog (the length of the **leash)** ≤ ***r***.
- Frechet**(P,Q,r)=YES** if the answer is positive for all time stamps.
- (if not, a longer leash is need. If yes, maybe a shorter one is sufficient.
- So we could use binary search.

Computing Frechet(P,Q,r)

```
Frechet(P,Q,r)
// c[1..n, 1..n] –  boolean array
// c[i,j]= Frechet(P[1..i],Q[1..j ],  r  )

Init:
c[1,1]=  ($\| p_1 – q_1 \| \le r$ ) (YES/NO)
For i=2 to n  c[i,1]=  ($\| p_i – q_1 \| \le r$ ) AND c[i-1,1]  (YES/NO)
For j=2 to n c[1,j]=   ( $\| p_1 – q_j \| \le r$ ) AND c[1,j-1]
```

Computing Frechet (P,Q,r) (cont.)

```
// c[1..n, 1..n] –  boolean array

Init- previous slide

For i=2 to n
    For j=2 to n
        c[i,j] = ( $\| p_i - q_j \| \le r$) AND
        {     c[i-1,j-1], // both jumps
         OR          c[i-1, j ] , // person jumped from $p_{i-1}$ to $p_i$ , dog stays at $q_j$
         OR          c[ i,j-1] . // person stayed at $p_i$ , dog jumped from $q_{j-1}$ to $q_j$.
        }
Return c[n.n]
```

Note – this is only the cost (that is the distance itself. We still need to find what is the series of steps that yield this cost

Comments

- This is actually the **Discrete** Frechet Distance (only distances between vertices counts). We do not discuss the **continuous** version.
- This is only the Decision problem – we actually want the shortest leash. We could use a binary search to approximate it. Exact algorithm outside the scope of this course
- If person/dog could move backward, the problem is called the **weak** Frechet.

Maurice René Fréchet

Dynamic-programming hallmark #1

(we saw this slide already)

Optimal substructure
An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If $z = \mathrm{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.

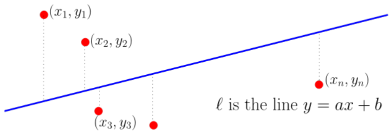
Dynamic-programming hallmark #2

Overlapping subproblems
A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Another application of DP: **Clustering**

(source: Kleinberg & Tardos 6.3)



ℓ is the line $y = ax + b$

- Given points $P = \{(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)\}$ find a line minimizing $Err(\ell, P)$

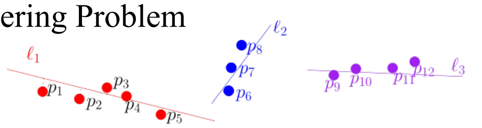
The fitting error

- $$Err(\ell, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$
  that is, the sum of squares of vertical distances from each (x_i, y_i) to ℓ.
- Solution

$$a = \frac{n\,\Sigma\, x_i y_i - (\Sigma\, x_i)(\Sigma\, y_i)}{n\,\Sigma\, x_i^2 - (\Sigma\, x_i)^2}$$

$$b = \frac{\Sigma\, y_i - a\,\Sigma\, x_i}{n}$$

Clustering Problem



Given a point set $P = \{p_1 \dots p_n\}$ sorted from left to right, and a cluster penalty $R > 0$.
Problem: Find a partition of P into **runs** (***clusters***)

$$(p_1, p_2 \cdots p_{i_1}), (p_{i_1+1}, p_{i_1+2} \cdots p_{i_2}), \dots, (p_{i_{k-1}+1}, p_{i_{k-1}+2} \cdots p_n)$$

and lines $\ell_1 \dots \ell_k$ such that the **total clustering cost,** $tct(\{p_1 \dots p_n\})$ is as small as possible. We define the **total clustering cost,** $tct(\{p_1 \dots p_n\})$ the sum of k penalties $(k \cdot R)$, plus the sum of the fitting errors between the points in each cluster and the line the fit them best;

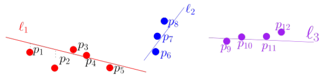
$tct(\{p_1 \dots p_n\})$ is defined as the value

$$R + Err(\ell_1, \{p_1, p_2 \dots p_{i_1}\}) + \quad R + Err(\ell_2, \{p_{i_1+1}, p_{i_1+2} \cdots p_{i_2}\}) + \dots + R + Err(\ell_k, \{p_{i_{k-1}+1}, p_{i_{k-1}+2} \cdots p_{i_1}\})$$

Note that if $R = 0$ (no penalty on new clusters) then the optimum clustering uses $\frac{n}{2}$ runs:
$(p_1, p_2), (p_3 \dots p_4), \dots, (p_{n-1} \cdots p_n)$. If R is huge, then the opt uses only one cluster, containing all the points.

In the example on top, $k = 3,\ i_1 = 5,\ i_2 = 8$

Algorithm



- Preprocessing: for every pair of i and j (where $j < i$) compute the line $\ell_{j,i}$ that best fit the points $\{p_j, p_{j+1}, p_{j+2} \cdots p_i\}$. Store in a table the value $e[j, i] = Err(\ell_{ji}\ , \{p_j, p_{j+1} \cdots p_i\})$
- Let c[i] = cost of the cost of the opt clustering of the points $\{p_1 \dots p_i\}$. This term includes both the sum of errors and the sum of penalties. At the i'th step of the algorithm, we assume that $c[0], c[1], c[2], \dots c[i-1]$ are already computed, and using these values, we will compute ***c[i].***
- We will also use an array $\Pi[1..n]$ its role is similar to the value $\Pi[v]$ in Dijkstra alg'.

Algorithm:

```
1. Init: c[0]=0 ; $c[i] = \infty$ and $\Pi[i] = NULL$,  for every $i > 1$  ; $\Pi[i] = NULL$
2. For i=2 to n do {
  2.1.For j=0 to i-1
  2.2. If  $c[i] > c[j] + R + e[j+1,i]$  then
      2.2.1.$c[i] = c[j] + R + e[j+1,i]$
      2.2.2.$\Pi[i] = j$ //The rightmost point in the previous cluster.
3. Return c[0]=n
```

Idea: p_i must belong to a cluster. We pay R for this cluster. The inner loop finds what is the best point p_{j+1} to be the leftmost point of this cluster.

Summarizing

- The algorithm takes $O(n^3)$ and $O(n^2)$ space
- (for preprocessing $d[j,i]$)
- Note – we did not discuss how to reconstruct the solution itself. We only calculated its cost