CSc445 Algorithms

Everything you always wanted to know about Quick Sort,

What lessons could QuickSort teaches us about other algorithms

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Based on slides curacy of

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QuickSort –
example of the
divide-and-concourse paradigm

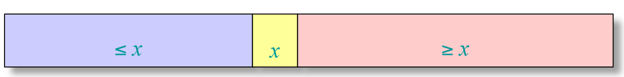
- Proposed by C.A.R. Hoare in 1962.
- Sorts "in place" (no need for extra space). Like insertion sort, but not like merge sort.
- Very practical (with tuning).

Divide and conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.
2. ***Conquer:*** Recursively sort the two subarrays.

- ***Combine:*** Trivial.



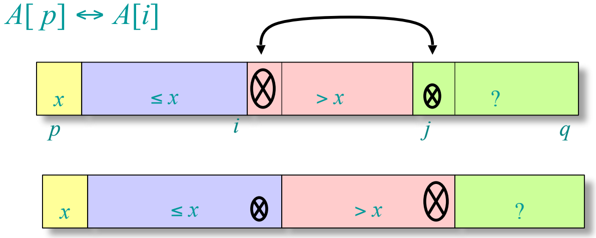
Key: *Linear-time partitioning subroutine.*

Partitioning subroutine

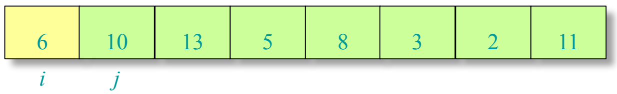
```
PARTITION(A, p, q)  ▻ A[ p . . q]
    x ← A[ p]           ▻ pivot = A[ p]
    i ← p
    for j ← p + 1 to q  ▻ j is hunting for small keys
        do if A[ j] ≤ x        ▻ Should send A[ j] to the left.
              then{
                   i ← i + 1          ▻ Now A[i]>x
                   exchange A[i] ↔ A[ j]  ▻ Fix A[i]>x
              }
    exchange A[ p] ↔ A[i]
    return i
```

Running time = $O(n)$ for n elements.

Invariant:



Example of partitioning



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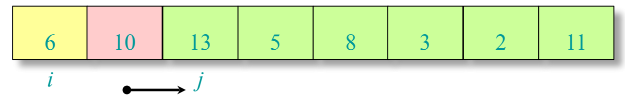

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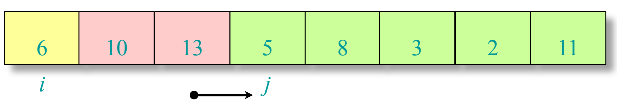

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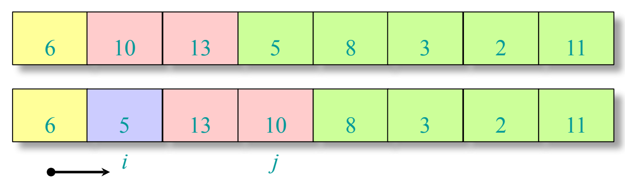

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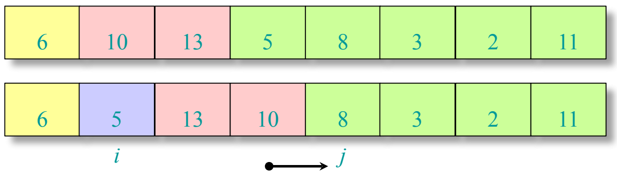

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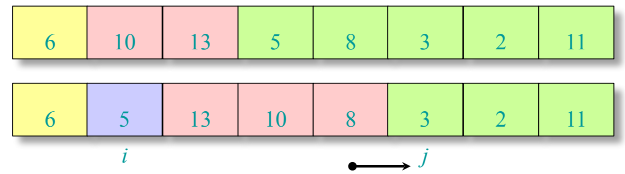

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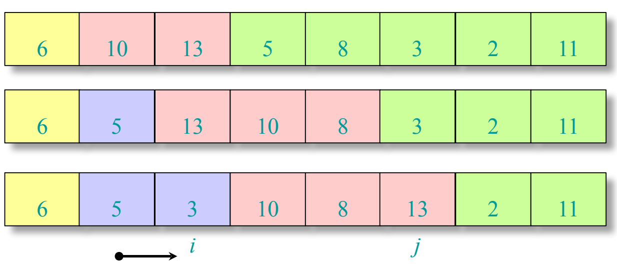

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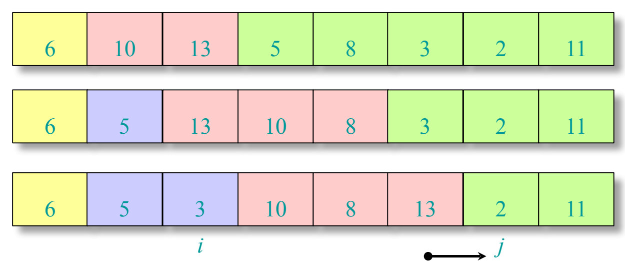

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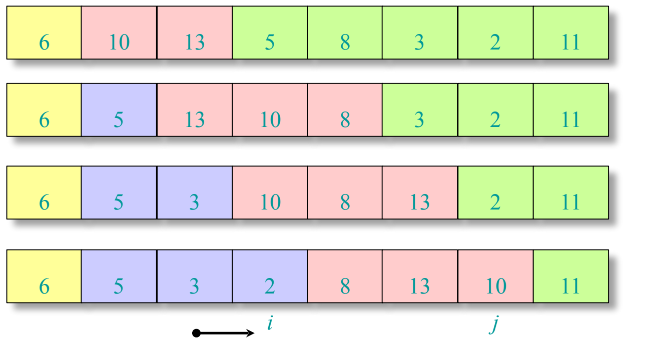

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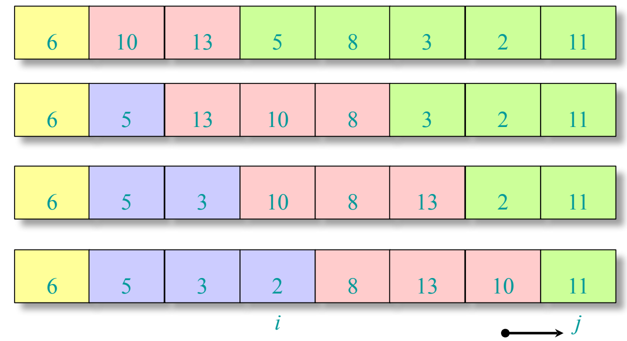

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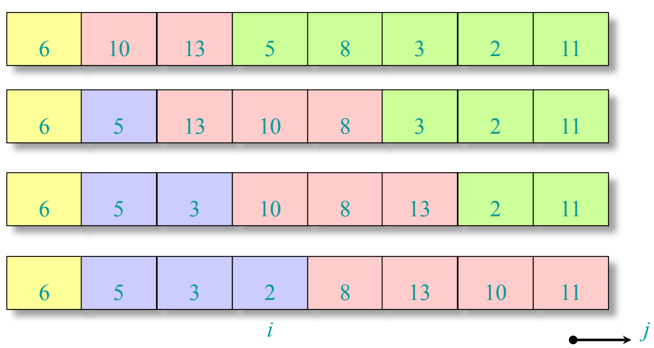

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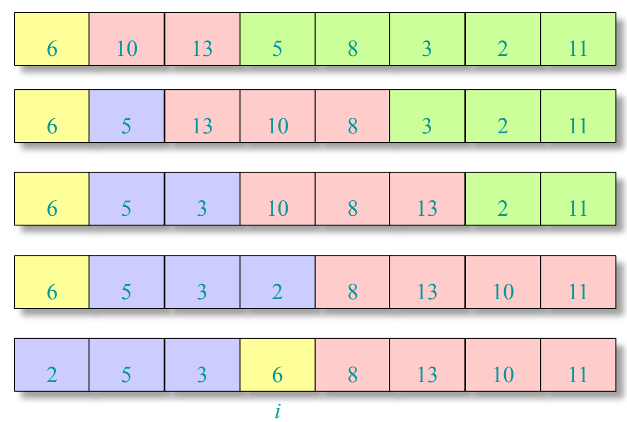

Pseudocode for quicksort

QUICKSORT(A, p, r)
    if $p < r$ //do something only if contains at least 2 keys
        then $q \leftarrow$ PARTITION(A, p, r) //both perform partition, and return index of pivot
            QUICKSORT($A, p, q-1$) //QS left part
            QUICKSORT($A, q+1, r$) //QS right part

Initial call: AUICKSORT(A, 1, n)

Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.

Worst-case of quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \quad \textit{(arithmetic series)} \end{aligned}$$

Worst-case recursion tree

$T(n) = T(0) + T(n-1) + cn$

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$T(n)$

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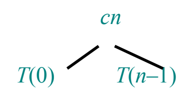

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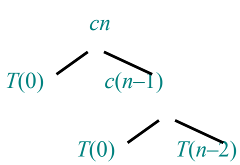

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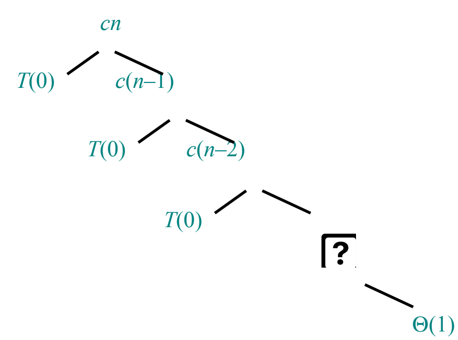

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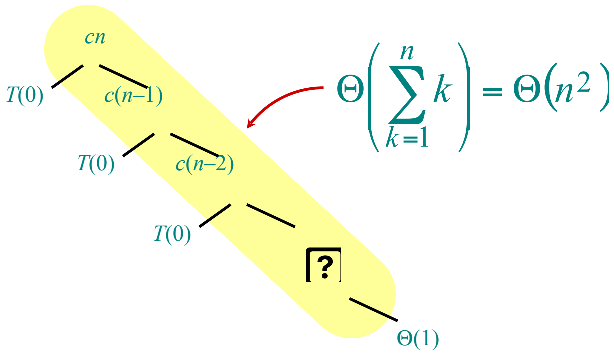

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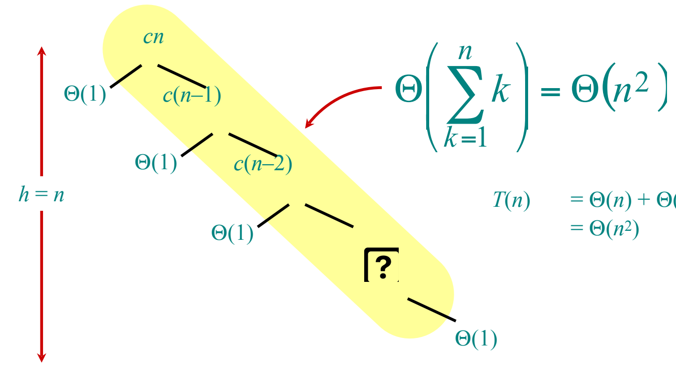


$$T(n) = \Theta(n) + \Theta(n^2) = \Theta(n^2)$$

Best-case and almost best-case analysis

If we are lucky, PARTITION splits the array evenly:

$$T(n) = 2T(n/2) + \Theta(n) = \Theta(n \lg n)$$

(same as merge sort)

What if the split is $\frac{1}{10} : \frac{9}{10}$?

That is, both sub-arrays contains at least 10% of the keys (possibly more)

$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

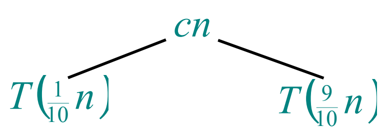
We call such a partition an **almost-optimal** partition.

What is the running time in this case?

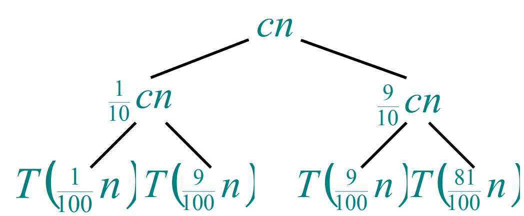
Analysis of "almost-best" case

$T(n)$

Analysis of "almost-best" case



Analysis of "almost-best" case



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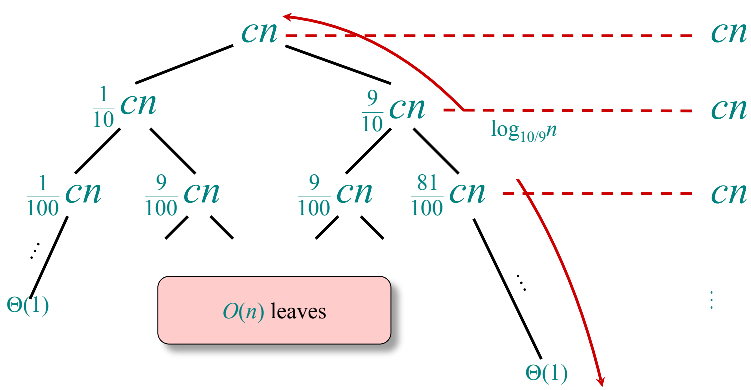

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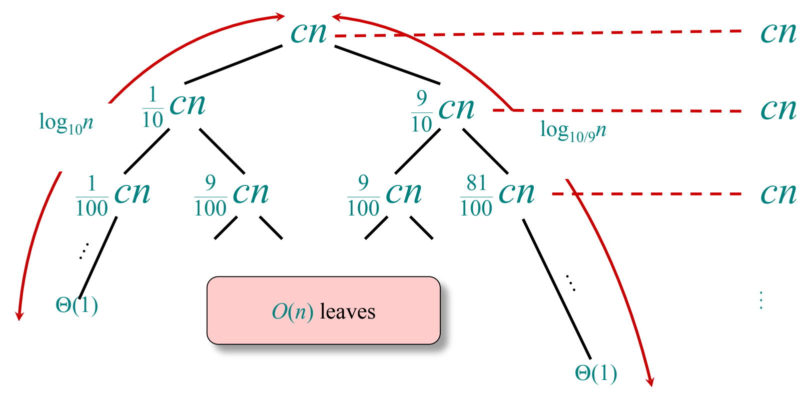


$cn \log_{10} n \leq$ $T(n) \leq cn \log_{10/9} n + O(n)$

$\leq 8\ n\ c \log_2 n$

QS needs O(n log n) if partition are almost optional

Each time the algorithm invested some work, it moves a key from one location to another

Consider a key x.

When the algorithm starts, it is in an array of size ***n***
Then x is shifted into an array of size. $\leq (0.9) \cdot n$
Next, x " " " " of size $\leq (0.9)^2 \cdot n$
Next, x " " " " of size. $\leq (0.9)^3 \cdot n$
⋮
After ***k*** times that x was shifted, its array's size $\leq (0.9)^{\mathbf{k}} \cdot n$

Max time that x is shifted:
$(0.9)^k n \leq 1$ *OR* $k \leq \log_{\left(\frac{10}{9}\right)} n \leq 8 \log_2 n = O(\log n)$
Next we need to multiply this number of the number of keys, yielding O(n log n)

Randomized quicksort

How can find a pivot that guarantees partitions with good ratios for *A[1..n]*, ?

We say that q is a **good pivot** for if

- at least 10% of the elements of *A[1..n]* are smaller than q, and
- at least 10% of the elements of *A[1..n]* are larger than q.

Best pivot: Pick the **median** of *A[1..n]*, as pivot.
(median – an element that is larger than half of the keys)
Then the time would obey $T(n) = cn+2T(n/2)$
Problem – need to work too hard to find the median (best pivot), so we will do with (only) a good pivot. (of course, we could first sort :-).)

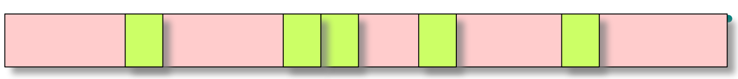
Finding a good pivot for *A[1..n]*

5-random-elements method. :

- Pick the **indices** of 5 elements at random from *A[1..n]*,
- *For k=1 to 5*

$$X[k] = A\left[\lfloor n \cdot \ rand() \rfloor\right]$$

A[1..n]



- Set q to be the median of *X[1..5]*

Finding a good pivot for *A[1..n]*

5-random-elements method. : Pick 5 elements at random from *A[1..n]*, and set q to be their median.

What it is the probability that q is **not** a good pivot ?

- Let S be the elements of *A[1..n]* which are the 10% smallest.
- The probability that an elements picked at random is in S is *0.1*.
- q is in S only if **at least 3** of the 5 elements that we pick are in S.
- The probability that this happens is

$0.1^5 +$ (all in S) $\quad 5 \cdot 0.1^4 \cdot 0.9 +$ (4 in S, one not in S) $\quad 10 \cdot 0.1^3 \cdot 0.9^2 =$ (3 in S)

$= 0.00001 + 0.00045 + 0.00810 = 0.00856$

- This is also the probability that q is in the 10% largest elements.
- In other words: with probability ≥98%, q is a good pivot.

Putting it together

- If we performed a partition which is **not** almost optimal, nothing dramatically bad happens, we just wasted some time. Each such partition takes linear time, but has no effect.

- However, each partition is, with probability $\geq 98\%$ is good, and we obtain an almost-optimal pivot.

- Hence the expected time of QuickSort (if the 5 random keys methods is used) is

  O(n log n) +0.02·O(n log n)=O(n log n)

Randomized quicksort – cont
Finding good pivots

Putting it together, during QS, each time that we need to find a pivot, we use the "5 random elements" method.

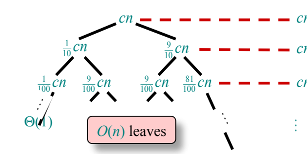
With probability 98%, we find a good pivot.

The overall time that we spend on good partitions is much smaller than the time we spent on bad partitions.

(note – bad partitions are not harmful – they are just not helpful)

So the recursions formula $T(n) = cn + T(n/10) + T(n\, 9/10)$ still apply, leading to running time $O(n \log n)$.

This is expected running time – there is a chance that the actual running time is $\Theta(n^2)$, but the probability that it happens is very slim.



Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort behaves well even with caching and virtual memory.

Median Selection

- (CLRS Section 9.2, page 185).
- For $A[1..n]$ (all different elements) we say that the rank of x is i if exactly $i-1$ elements in A are smaller than x.
- In particular, the median is the $\lfloor n/2 \rfloor$-smallest.
- To find the median, we could sort and pick $A[\lfloor n/2 \rfloor]$ (taken $O(n \log n)$).
- We can do better.

Median Selection-cont

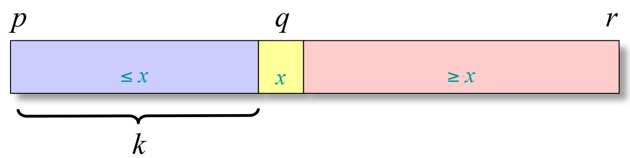
RS(*A, p, r, i*){

//**R**andomize **S**election: Returns *i*'st smallest element in *A[p..r]*.

//Assumption: Input is valid and elements are different.

- If *p==r* return A[*p*]
- *q*=PARTITION(*A,p,r*) ;
  - //Partition using the 5-random element method
- *k=q-p*
- If *i==k+1* return *A[q]*
- If *i<k* return RS(*A, p, q-1, i*) // Note the difference from QS
- Else return RS(*A, q+1, r, i-k-1*)

}



Time analyis

- Recall: With high probability, we pick a good pivot:
  - Not in the 10% smallest or largest:
- Hence, we get rid of at least 10% of the elements of *A*
- So, $T(n)=cn+T(0.9\,n)$.
  - $T(n)=c(n+0.9n+0.9^2n+0.9^3n+\ldots) =$
  $cn(1+0.9+0.9^2+0.9^3+\ldots) =$
  $cn(1/(1-0.9)) = O(n)$.
- So the expected time is linear. (yuppie)

As in the case of QS, partitions which are not good are not harmful, just not helpful.