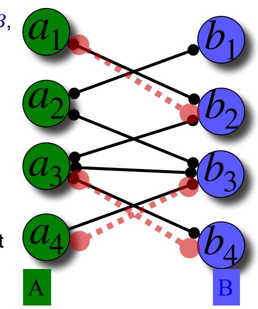
cs445

Bipartite Matching and Max-Flow in a Network

Alon Efrat

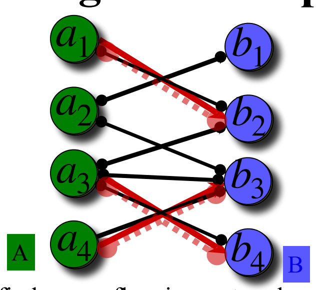
Application: Bipartite Matching.

- A graph $G(V,E)$ is called **bipartite** if V can be partitioned into two sets $V=A\cup B$, and each edge of E connects a vertex of A to a vertex of B. We sometimes denote these graphs by $G(A\cup B,E)$
- Example: The set $A=\{a_1 \ldots a_n\}$ is a set of instructors, the set $B=\{b_1 \ldots b_n\}$ is the set of courses. There is an edge $(\mathbf{a_i}, \mathbf{b_j}) \in \mathbf{E}$ iff instructor a_i could teach course b_j
- A **matching** is a set of edges M of E, where each vertex of A is adjacent to at most one vertex of B, and vice versa.
- *(in the example, each instructor will teach at most one course, and vice versa)*
- **Maximum-cardinality matching:** *Find a matching with as many edges as possible*
- This problem could be solved with in O(nm) time using Ford-Fulkerson algorithm. Faster algorithms exist as well. However, we will use it as an example to the ease of using ILP.



In red: Edge of the matching

Matching and flow problem



- If we know how to find a max-flow in a network, we could use it so solve a matching problem: For this, we need to express the matching problem as a flow problem:
  A. Add a vertex s, and connect it to each vertex of A.
  B. Add a vertex t, and connect each vertex of B to t.
  C. Assign capacity of 1 to each edge (u,v).
- Find max flow. Assume it is an **integer** flow, so the flow across each edge is either 0 or 1
- Each edge of G that carries flow is in the matching.
- Each edge of G that **does not** carry flow is **not in** the matching.
- <u>**Claim:**</u> The edge between A and B that carry flow form a matching M.
- Proof: We just need to show that no instructor a_i is matched to two courses b_j, b_k, and vice versa

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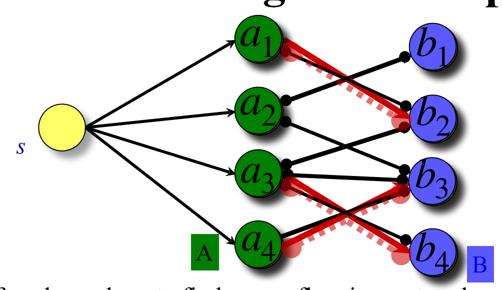

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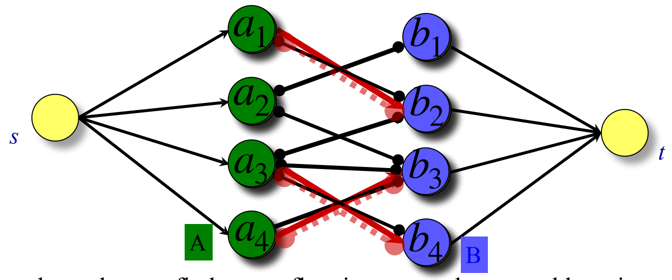

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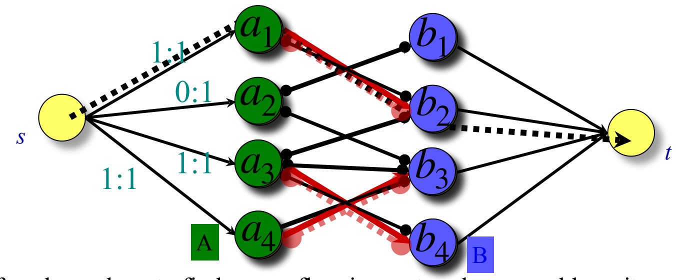

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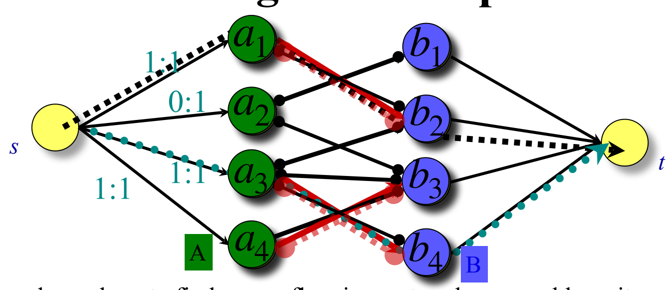

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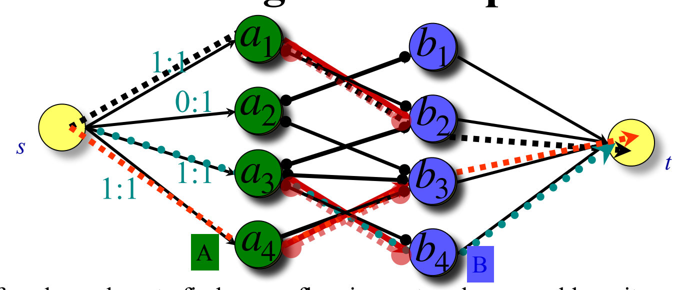

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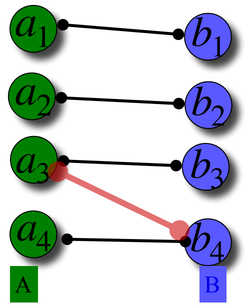
Ford-Fulkerson algorithm for finding max bipartite matching

- This algorithm is actually appropriate for any network flow problem, but notations and proofs are simpler if we concentrate on the matching directly.

- Algorithm: Start then $M = \emptyset$. No edge is in the matching.
- Output: $|M|$ is as large as possible
- At each step of the algorithm, we increase the cardinality of M by 1.

- General Step: Assume M is given. Terminology:
- A ***matched vertex*** is a vertex which is an edgepoint of an edge of M. Vertices that are not matched are called ***exposed vertices***.

- We will denote all matched edge $M \subseteq E$ by a thick red segments, and edge of $E \setminus M$ are depicted by a straight edge. Sometimes we will denote these edges by 



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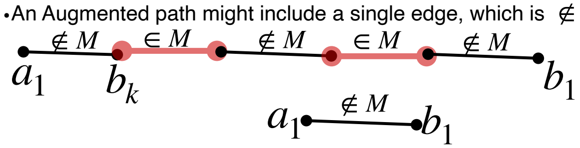
In red: Edge of the matching

- An **augmenting path** is a path that starts with an expose vertex of A, ends at an exposed vertex of B, and its edges alternates: An edge $\notin M$ followed by an edge $\in M$, followed by an edge $\notin M$ and so on.

- An Augmented path might include a single edge, which is $\notin M$



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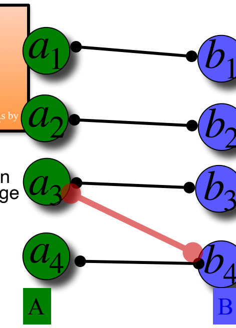

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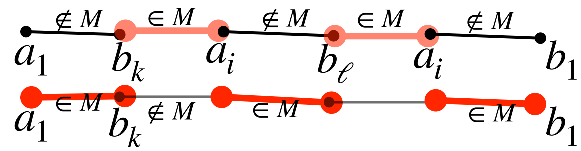
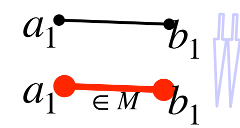
- Augmented path might include a single edge, which is $\notin M$
- Lets p be an augmenting path. The operation of **augmentation a path** consists of
  - Insert into M all edges of p which are $\notin M$, and remove from M all edges that originally were in M.



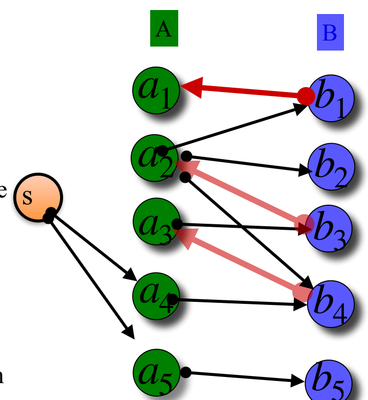


Claims:

1. A vertex that was matched before the augmentation, is matched after the augmentation
2. Matching is 1-1 (no course taught by two teaches, no teacher teaches two courses.
3. Augmentation increases the number of edges in the matching by 1.

How to find augmenting paths

- Makes the graph a directed graph:
  - Edges $\in M$ are directed from right to left
  - Edges $\notin M$ are directed from left to right
  - Add a vertex s, and connect it to every **exposed** $a_i \in A$
- Run DFS or BFS from s.
- Every path that leads to an exposed vertex must be an augmented path. And
- If there is an augmented path, this process will find this path.



Once an augmented bath is found, we augment its edges, and restart (re-bulding the directed graph).

If no augmented path is found, stop - M is maximum cardinality matching. (we will need to prove it)

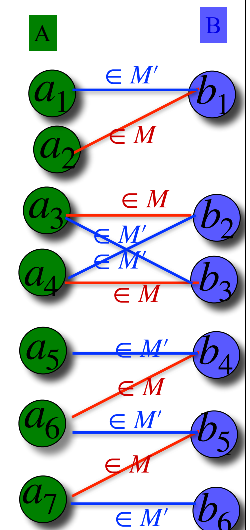
Running time: Each iteration, we increase |M| by 1, so the number of iterations is $\leq \min\{|A|,|B|\} \leq n$.

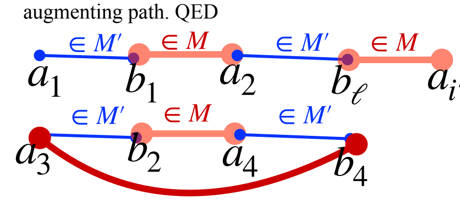
- Finding an augmented path is done via DFS or BFS, so its time is $O(|E|+|V|)$

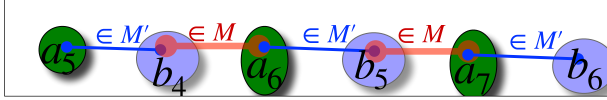
- Overall time $O(|E||V|) = O(mn)$

Optimality Theorem : M is maximum iff there is no augmenting path
Proof: One direction is trivial: If there is an augmenting path, then we could increase $|M|$, so M it is not optimum. Lets prove the second direction:

1. On the other hand, assume M is not optimum. Let M' be another matching such that $|M| < |M'|$.
2. Let think about $U \stackrel{def}{=} M \oplus M' \subseteq E$. These are the edges which are either in M or in M', but not in both. Some edges of E are in neither M nor in M'.
3. Each vertex $v \in V$ is on $\leq$ one edge of M and on $\leq$ one edge of M'.
4. Every path of U is an alternating path - an edge from M followed by an edge from M' and so on.
5. U might consists of several pathS and several cycleS.
6. Every cycle must have an even length (why?).
7. However, since |M|<|M'|, one of the alternating path contains more edges from M'. This must be a path whose first and last edge are from M'. This is an augmenting path. QED







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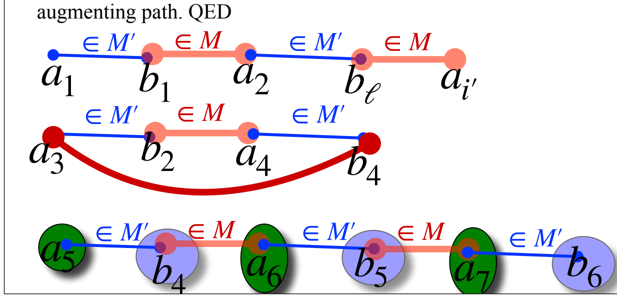

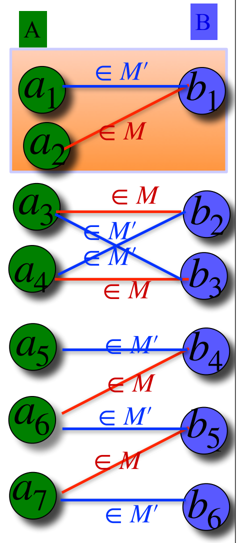

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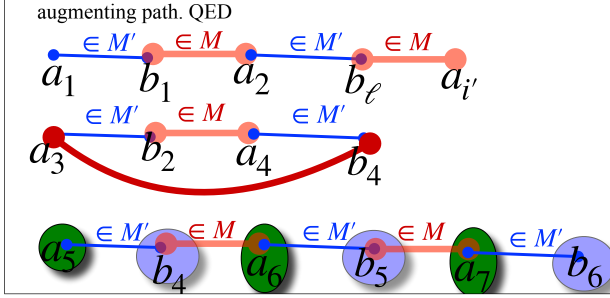

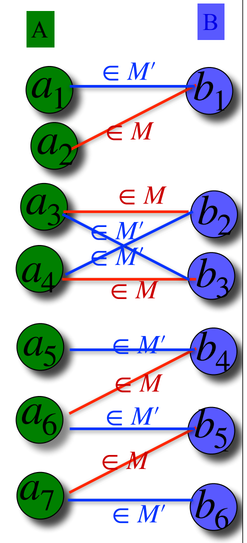

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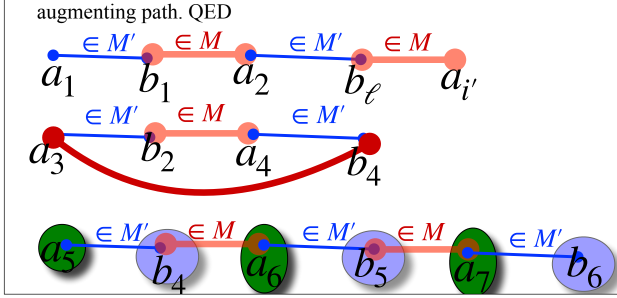

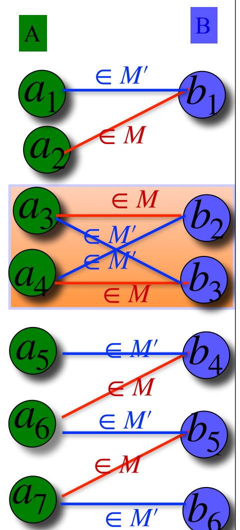

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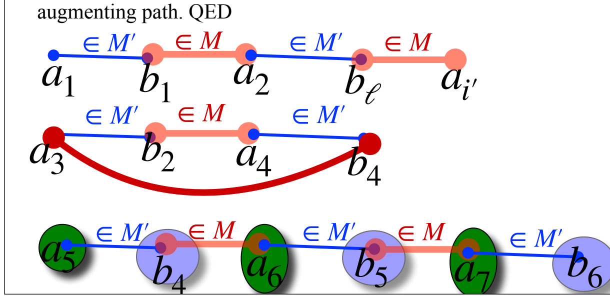

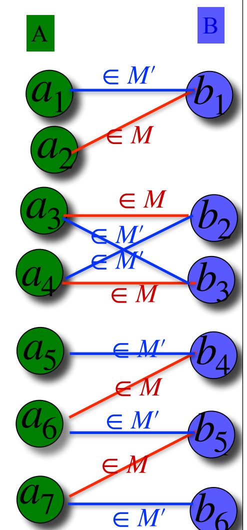

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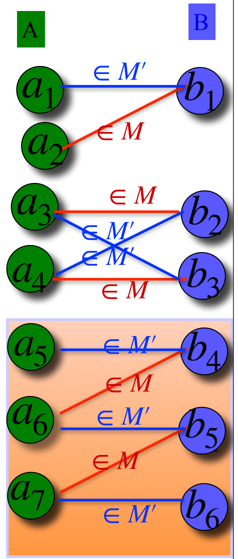

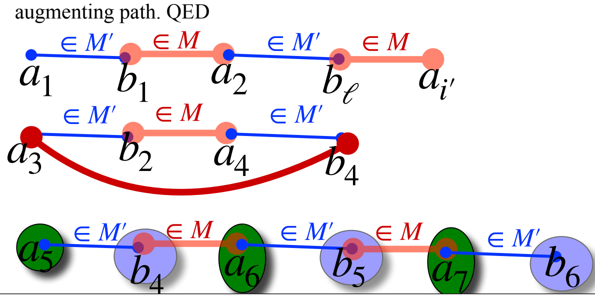

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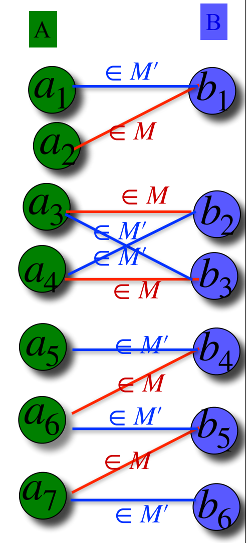
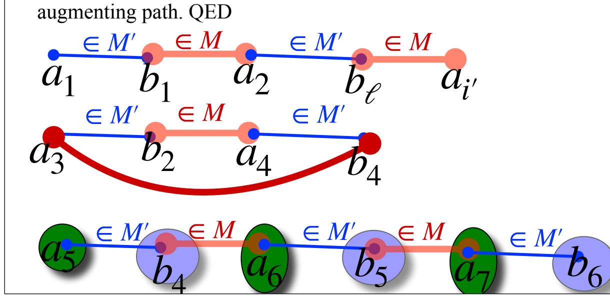
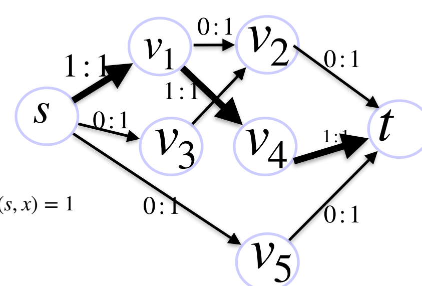




Max Flow in 0/1 Network

- A 0/1 network is a directed graph G(V,E), where there are given special vertices $s,t \in V$, and the capacity of every edge is 1. (instead of $c(u,v)$)
- A flow is **legal** if
  - for every edge $(u,v) \in E$ we are given the flow $f(u,v)$ across the edge (u,v).
  - $0 \leq f(u,v) \leq 1$. (capacity constrains)
  - For every vertex $v \in V - \{s,t\}$ we have
  - $\sum_{(w,v)\in E} f(w,v) = \sum_{(v,x)\in E} f(v,x)$ Flow conservation
- The value of the flow is $|f| := \sum_{(s,x)\in E} f(s,x)$ flow from s. This is the value we want to **maximize**.
- The gaol is to maximize the value of the flow.
- The matching problem is a special case of this problem.

(in this example) $|f| := \sum_{(s,x)\in E} f(s,x) = 1$



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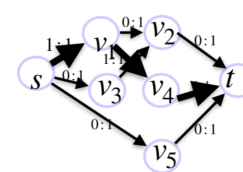
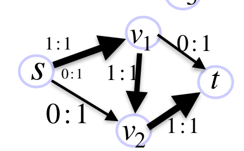
- When we solve this problem using LP, we might find solutions that are non-integers.
- We can use ILP. Sometimes very efficient. Sometimes very slow.
- Ford-Fulkerson Algorithm: A sequence of iteration, at each, the value of the flow, |f| will be increased by 1.
- Under this algorithm, the flow across every edge is either 0 or 1. (but never 0.5)
- A greedy approach would be: Find a path $s \rightsquigarrow t$ of edges that carry zero flow. Increase the flow along this path, and repeat.
- This approach might not work (we saw a similar example in matching). Heavy edges carry flow.





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- if f(u,v)=0 then $(u,v) \in E_f$.
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Ford-Fulkerson Algorithm:

1. Assume that some 0/1 (legit) flow f is given.
2. Create a new graph $G_f(V, E_f)$. In the textbook, it is called the **residual network**.
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   - If $(u, v) \in E$ (edge not reversed) then $f(u, v) = f(u, v) + 1$
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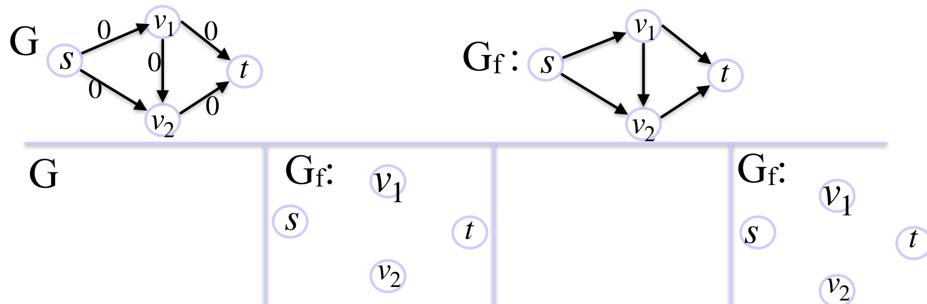

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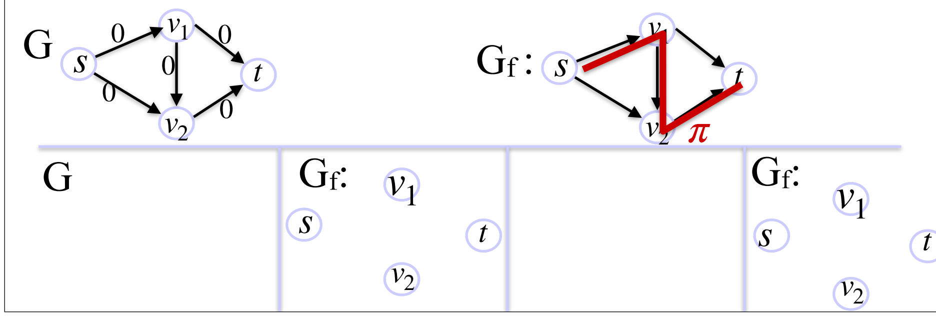

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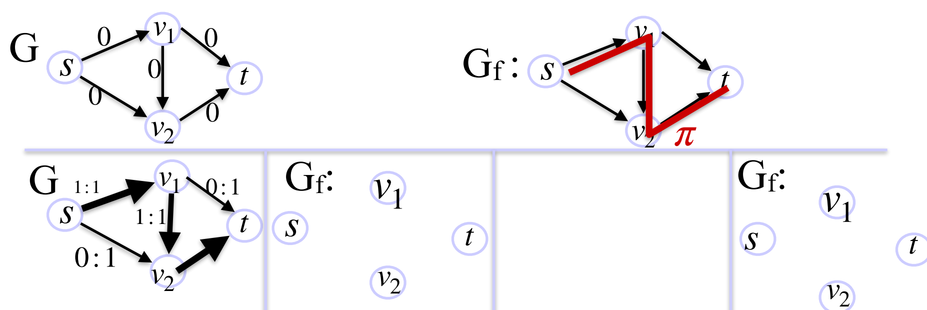

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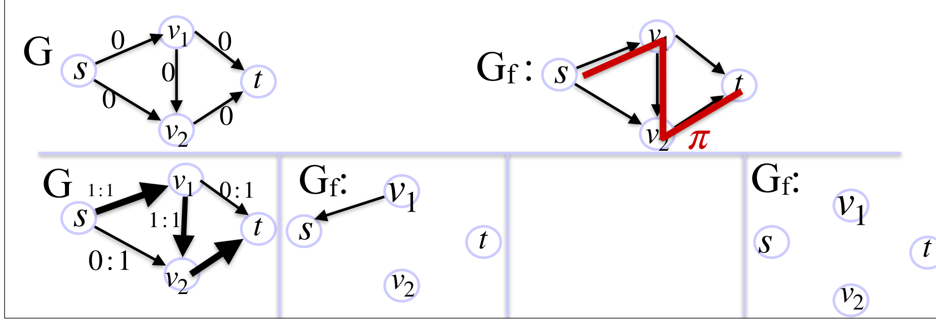

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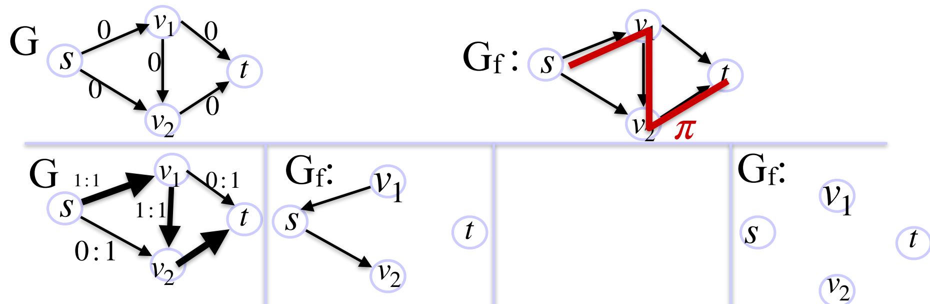

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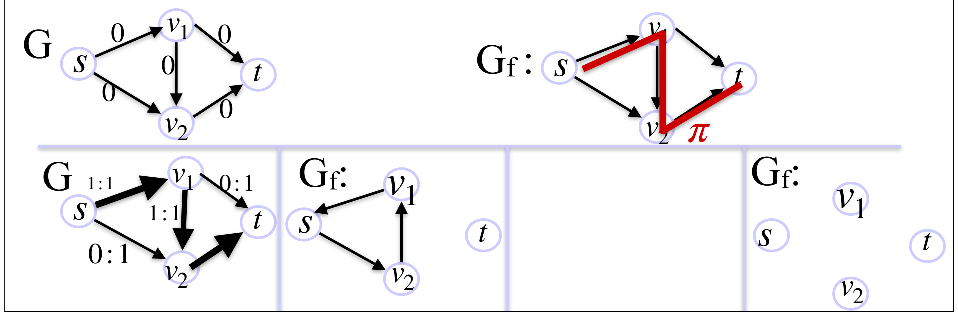

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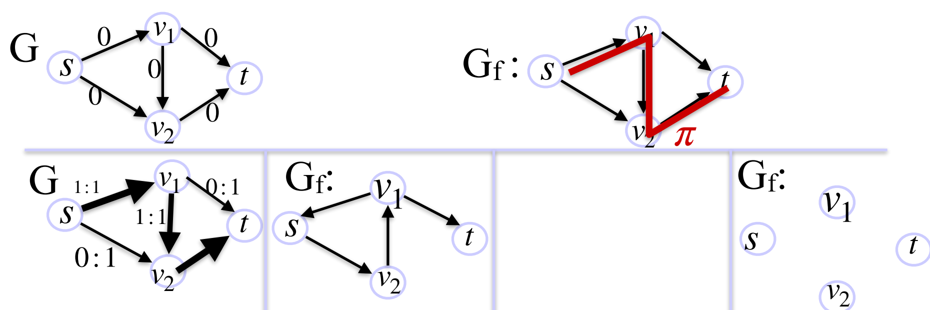

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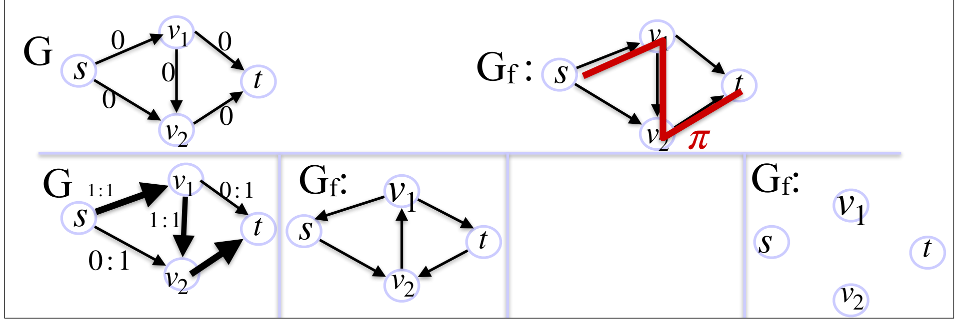

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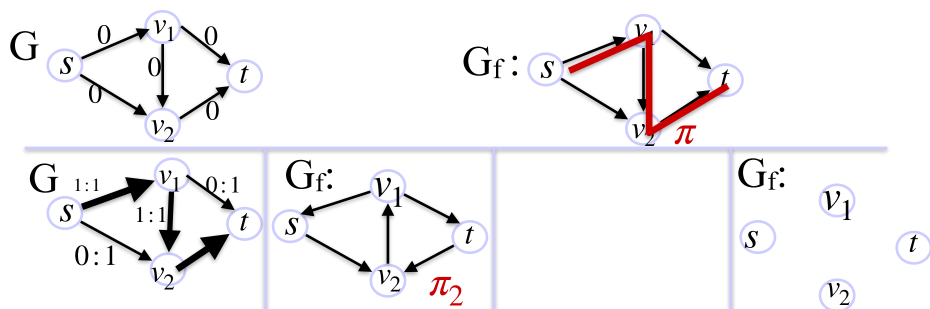

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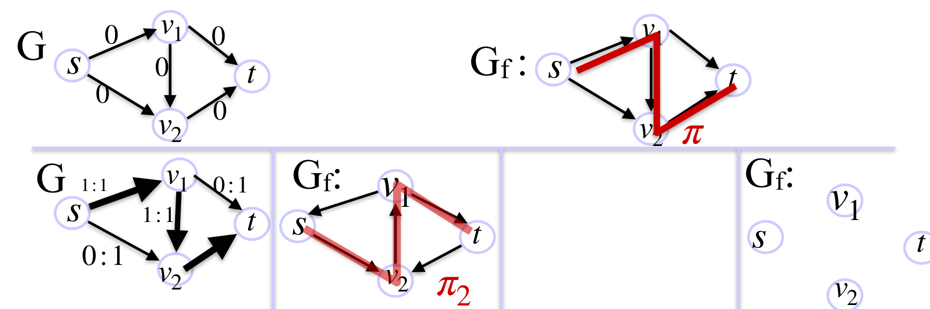

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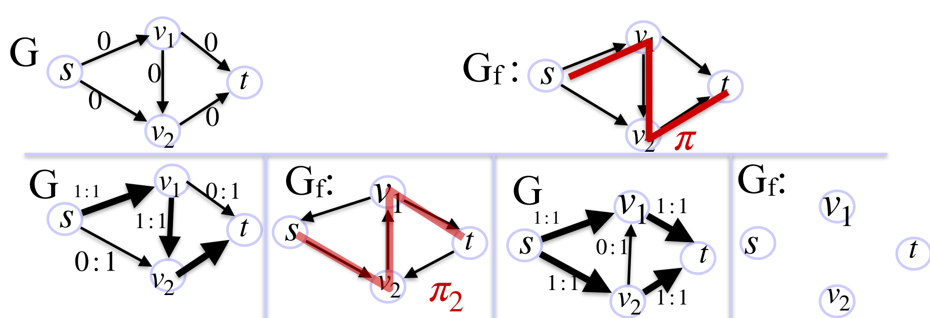

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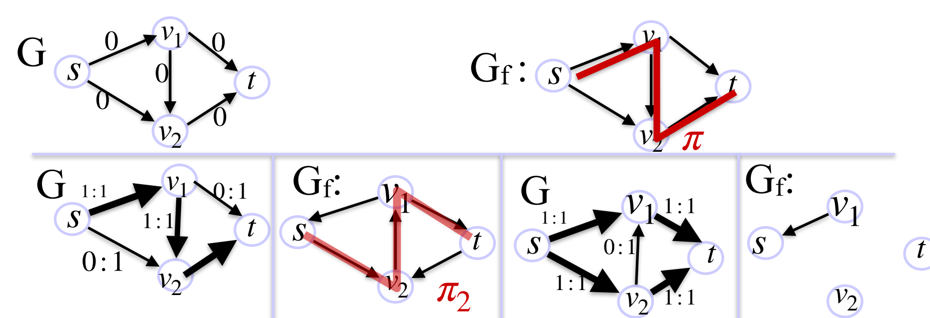

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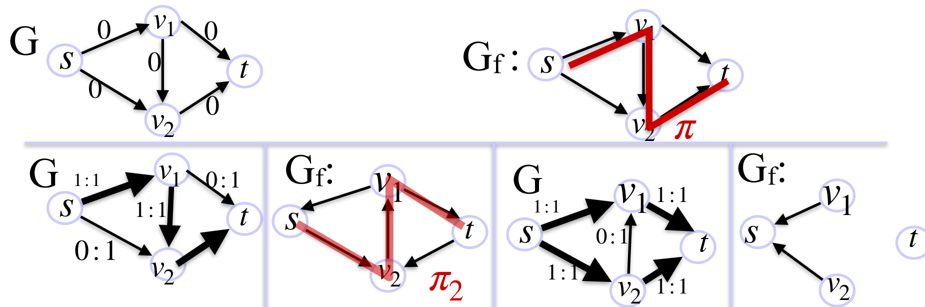

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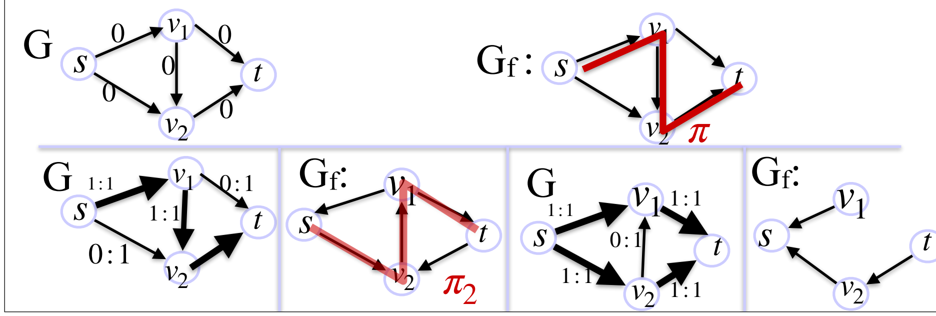

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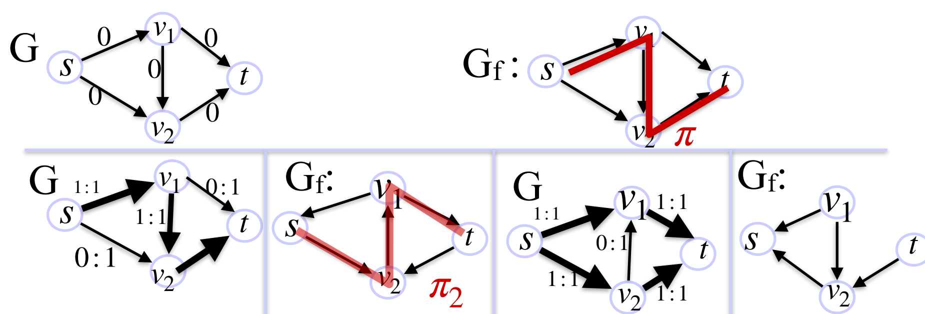

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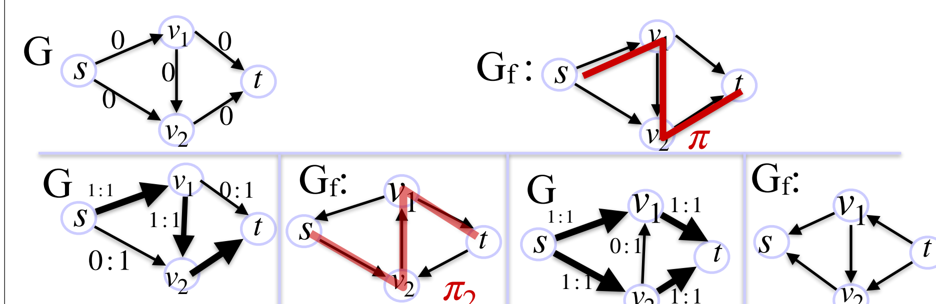
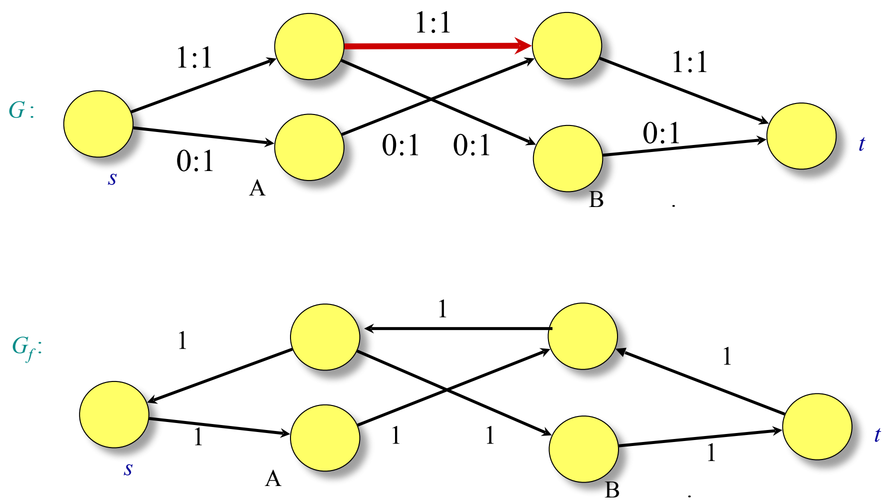


Example – maximum matching



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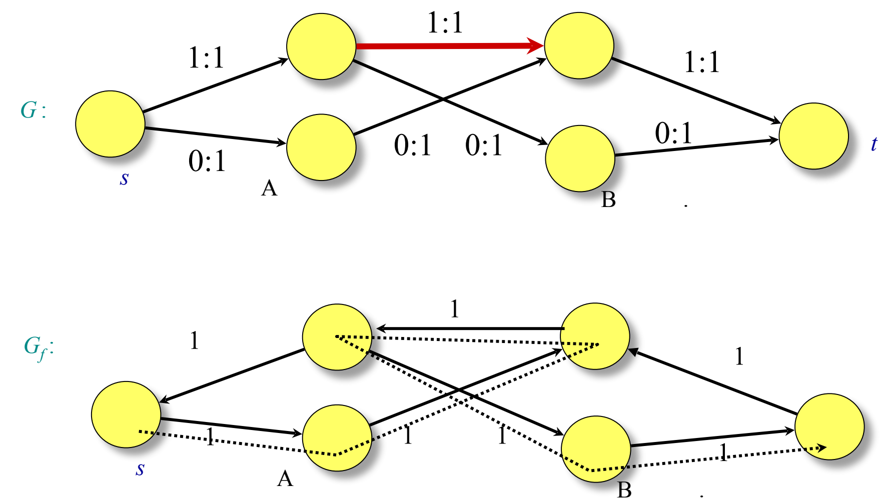
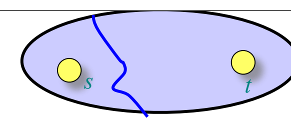


Ford-Fulkerson max-flow algorithm

- Start: $f[u, v] \leftarrow 0$ for all $(u, v) \in E$
- While **(1) {**
  - construct G_f
  - if an augmenting path p in G_f exists then
    augment f //Any path would do
  - else exit }

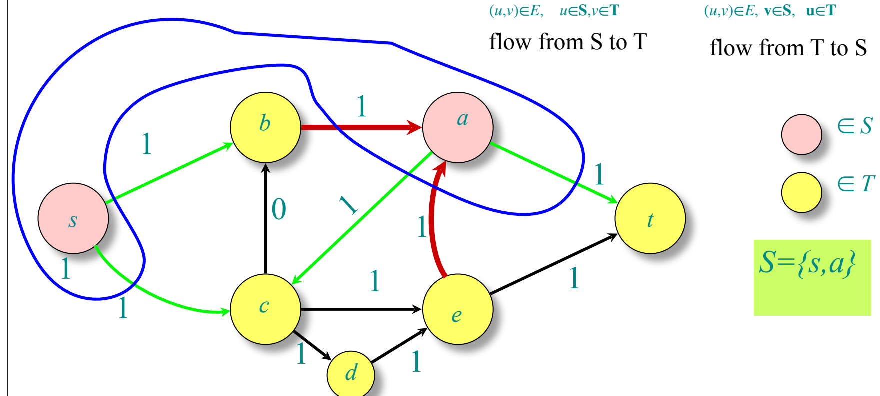
Cuts



Definitions. A ***cut*** $(\mathbf{S}, \mathbf{T})$ of a flow network $G=(V, E)$ is a partition of V such that $s \in \boldsymbol{S}$ and $t \in \boldsymbol{T}$.

If f is a flow on G, then the ***flow across the cut*** denoted $f(\mathbf{S}, \mathbf{T})$ is

$$f(\mathbf{S},\mathbf{T}) := \underbrace{\sum_{(u,v)\in E,\ u\in \mathbf{S}, v\in \mathbf{T}} f(u,v)}_{\text{flow from S to T}} - \underbrace{\sum_{(u,v)\in E,\ v\in \mathbf{S},\ u\in \mathbf{T}} f(u,v)}_{\text{flow from T to S}}$$



$S=\{s,a\}$

$f(\mathbf{S},\mathbf{T}) = f(s,b) + f(s,c) + f(a,t) + f(a,c) - f(b,a) - f(e,a)$

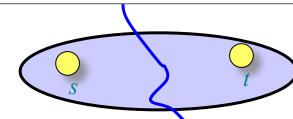
Lemma: Flow across the cut

Remember $|f| := \sum_{(s,v)\in E} f(s,v)$ That is, it is the flow leaving s

Lemma. For any flow f and any cut $(\mathbf{S}, \mathbf{T})$, we have $|f| = f(\mathbf{S},\mathbf{T})$ (*flow from T to S*).

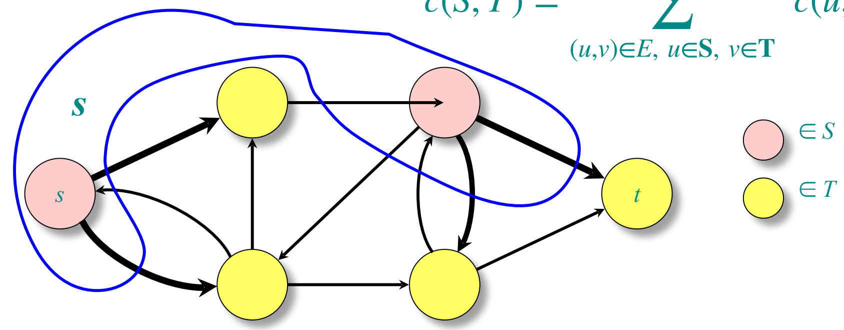
Proof: On whiteboard

Capacity of a cut



Definition. The ***capacity of a cut*** (S, T) is the number of edges across the cut

$$c(S,T) = \sum_{(u,v)\in E,\ u\in\mathbf{S},\ v\in\mathbf{T}} c(u,v)$$



Upper bound on the maximum flow value

Theorem. The value of any flow no larger than the capacity of any cut: $|f| \le c(S,T)$.

$$
\begin{aligned}
|f| &= f(\mathbf{S},\mathbf{T}) \\
&= \sum_{(u,v)\in E,\ u\in\mathbf{S}, v\in\mathbf{T}} f(u,v) - \sum_{(u,v)\in E,\ v\in\mathbf{S},\ u\in\mathbf{T}} f(u,v) \\
&\le \sum_{(u,v)\in E,\ u\in\mathbf{S}, v\in\mathbf{T}} f(u,v) \\
&\le \sum_{(u,v)\in E,\ u\in\mathbf{S}, v\in\mathbf{T}} c(u,v) \qquad = \mathbf{C}(\mathbf{S},\mathbf{T})
\end{aligned}
$$

.

Max-flow, min-cut theorem

Theorem. The following are equivalent:

1. $|f| = c(S, T)$ for some cut (S, T).
2. f is a maximum flow.
3. f admits no augmenting paths.

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Proof.

(1) ⇒ (2): Since $|f| \le c(S, T)$ for any cut (S, T) (by the theorem from a few slides back), the assumption that $|f| = c(S, T)$ implies that f is a maximum flow.

(2) ⇒ (3): If there were an augmenting path, the flow value could be increased, contradicting the maximality of f.

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min-cut

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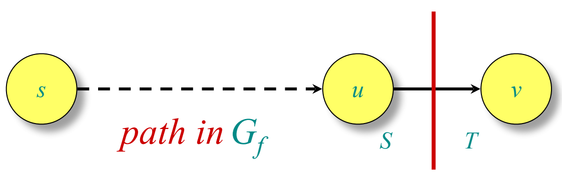
(2) ⇒ (3): If there were an augmenting path, the flow value could be increased, contradicting the maximality of f.

(3) ⇒ (1): Define $S = \{v \in V \mid$ there exists a path in G_f from s to $v\}$,

Let $T = V - S$. Since f admits no augmenting paths, there is no path from s to t in G_f.
Hence, $s \in S$ and $t \notin S$, So $t \in T$.

Thus (S, T) is a cut.



Consider edge (u,v) $u \in S$, $v \in T$. Observe that f(u,v)=1, since if it was zero, we would add (u,v) to the path.

Thus, $f(u, v) = c(u, v)$

Summing over all $u \in S$ and $v \in T$ yields $f(S, T) = c(S, T)$, and since $|f| = f(S, T)$, the theorem follows.