Touring a Sequence of Polygons

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Problem:

Given a sequence of $k$ polygons in the plane, a start point $s$, and a target point, $t$, we seek a shortest path that starts at $s$, visits in order each of the polygons, and ends at $t$. 
Related Problem: TSPN:

If the order to visit \( \{P_1, P_2, \ldots, P_k\} \) is not specified, we get the NP-hard TSP with Neighborhoods problem.

TSPN: \( O(\log n) \)-approx in general

\( O(1) \)-approx, PTAS in special cases
The Fenced Problem:

Here that part of the path connecting $P_i$ to $P_{i+1}$ must lie inside a simple polygon $F_i$, called the fence.
Applications: Safari Problem:

Previous result — $O(n^3 \log n)$. [Tan 2001]
Applications: Zookeeper Problem:

Optima result — $O(n \log n)$ [Bespamyatnikh 2001].
Applications: Watchman Route Problem:

Previous result — $O(n^4 \log n)$ [Tan, Hirata and Inagaki 99].

Fact: The optimal path visits the essential cuts in the order they appear along $\partial P$. 
Summary of Results:

- Disjoint convex polygons:
  \[ O(kn \log(n/k)) \] time, \( O(n) \) space to find \( \pi_k(t) \).

- Full combinatorial map: worst-case size \( \Theta((n - k)2^k) \)
  Output-sensitive algorithm; \( O(k + \log n) \)-time shortest path queries.

- TPP for nonconvex polygons: \( \text{NP-hard} \)
  FPTAS, as special case of 3D shortest paths.
Applications:

- Safari: $O(n^2 \log n)$ vs. $O(n^3)$
- Watchman: $O(n^3 \log n)$ vs. $O(n^4)$
  
  floating watchman: $O(n^4 \log n)$ vs. $O(n^5)$

We avoid use of complicated path “adjustments” arguments

- Parts cutting: $O(kn \log(n/k)$)
Unconstrained TPP: Disjoint Convex Polygons:

Given: $s, t$, sequence of disjoint convex polygons $\{P_1, \ldots, P_k\}$
Goal: Find a shortest $k$-path, $\pi_k(t)$ from $s = P_0$ to $t$.
Local Optimality Conditions: Assume that optimal path bounces from $P_i$.

Can bounce from an edge (incoming angle = bouncing angle) or bounce from a vertex.

Conclusion: If we know the orientations at which shortest paths edges leave the vertices of $P_i$, we know from which edge/vertex of $P_i$ the last segment of $\pi_i(q)$ leaves $P_i$, (for every $q \in \mathbb{R}^2$).
Lemma: For any $t \in \mathbb{R}^2$ and any $i \in \{0, \ldots, k\}$, there exists a unique shortest $i$-path, $\pi_i(p)$, from $s = P_0$ to $t$.

Thus, local optimality is equivalent to global optimality.
General Approach: Build a Shortest Path Map:

SPM\(_k(s)\): a decomposition of the plane into cells according to the combinatorial type of a shortest \(k\)-path to \(t\)

Bad news: worst-case size can be huge:

**Theorem:** The worst-case complexity of SPM\(_k(s)\) is \(\Omega((n - k)2^k)\)
$s$

$2^i$

$2^{i+1}$
Good (?) news: worst-case size cannot be *bigger* than “huge”:

**Theorem:** The worst-case complexity of $SPM_k(s)$ is $O((n - k)2^k)$

Size $m_i$ satisfies $m_i \leq 2m_{i-1} + O(|P_i|)$.

**Output-sensitive algorithm to build SPM:**

**Theorem:** One can compute $SPM_k(s)$ in time $O(k \cdot |SPM_k(s)|)$, after which a shortest $k$-path from $s$ to a query point $t$ can be computed in time $O(k + \log 2^n)$.
Some facts:

**Lemma:** In the TPP for disjoint convex polygons \( \{P_1, \ldots, P_k\} \), each first contact set \( T_i \) is a (connected) chain on \( \partial P_i \).

**Lemma:** For any \( p \in \mathbb{R}^2 \) and any \( i \), there is a unique point \( p' \in T_i \) such that \( \pi_i(p) = \pi_{i-1}(p') \cup \overline{pp'} \).
The Last Step Shortest Path Map:

$S_i$ = the last step shortest path map, subdivision according to the **combinatorial type** of the last rays of shortest paths passing through points $p \in \mathbb{R}^2$

$S_i$ decomposes the plane into cells $\sigma$ of two types:

1. cones with an apex at a vertex $v$ of $T_i$, whose bounding rays are reflection rays from $v$, and $v$ is the source of cell $\sigma$

2. unbounded 3-sided regions associated with edge $e$ of $T_i$, classified as *reflection cells* or *pass-through cells*. $e$ is the source of cell $\sigma$
Using $S_i$ to find a shortest $i$-path to query point $q$:

- cell $\sigma$ rooted at vertex $v$ of $T_i$
  
  \[ \pi_i(q) \text{ is } \sigma \]

  recursively compute $\pi_{i-1}(v)$ (locate $v$ in $S_{i-1}$, etc)

- cell $\sigma$ rooted at edge $e$ of $T_i$
  
  \[ \pi_i(q) = \pi_{i-1}(q) \]

  so recursively compute shortest $(i - 1)$-path to $q$

- $\sigma$ is pass-through:  recursively compute shortest $(i - 1)$-path to $q'$, the reflection of $q$ wrt $e$
Using the Last Step Shortest Path Map:

Note that Locating $q$ in $S_i$ takes $O(\log |P_i|)$, hence:

**Lemma:** Given $S_1, \ldots, S_i$, $\pi_i(q)$ can be determined in time $O(k \log(n/k))$
**Algorithm:**

Construct each of the subdivisions $S_1, S_2, \ldots, S_i$ iteratively:

For each vertex $v_j$ of $P_{i+1}$, we compute $\pi_i(v_j)$.

- If this path arrives at $v_j$ from the inside of $P_{i+1}$, then $v_j$ is not a vertex of $T_{i+1}$.
- Otherwise it is, and the last segment of $\pi_i(v_j)$ determines the rays $r_{ib}^i(v_j)$ and $r_{is}^i(v_j)$ that define the subdivision $S_{i+1}$.

**Theorem:** For a given sequence $\{P_1, \ldots, P_k\}$ of $k$ disjoint convex polygons having a total of $n$ vertices, a data structure of size $O(n)$ can be constructed in time $O(kn \log(n/k))$ that enables shortest $i$-path queries to any query point $q$ to be answered in time $O(i \log(n/k))$. 
TPP on Nonconvex Polygons:

**Proposition:** The TPP in the $L_1$ metric is polynomially solvable (in $O(n^2)$ time and space) for arbitrary rectilinear polygons $P_i$ and arbitrary fences $F_i$. The result lifts to any fixed dimension $d$ if the regions $P_i$ and the constraining regions $F_i$ are orthohedral.

**Theorem:** The touring polygons problem (TPP) is NP-hard, for any $L_p$ metric ($p \geq 1$), in the case of nonconvex polygons $P_i$, even in the unconstrained ($F_i = \mathbb{R}^2$) case with obstacles bounded by edges having angles 0, 45, or 90 degrees with respect to the $x$-axis.

**Open Problem:** What is the complexity of the TPP for disjoint non-convex simple polygons?