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Functional Programming

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Higher-Order Functions

Higher-Order Functions

- A function is **Higher-Order** if it takes a function as an argument or returns one as its result.
- Higher-order function aren't weird; the differentiation operation from high-school calculus is higher-order:

```
deriv ::  (Float->Float)->Float->Float
deriv f x = (f(x+dx) - f x)/0.0001
```

- Many recursive functions share a similar structure. We can capture such "recursive patterns" in a higher-order function.
- We can often avoid the use of explicit recursion by using higher-order functions. This leads to functions that are shorter, and easier to read and maintain.

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Currying Revisited I

- We have already seen a number of higher-order functions. In fact, any curried function of more than one argument is higher-order. Why? Well, when a curried function is applied to one of its arguments it returns a new function as the result.

— Uh, what was this currying thing? —

- A curried function does not have to be applied to all its arguments at once. We can supply some of the arguments, thereby creating a new specialized function. This function can, for example, be passed as argument to a higher-order function.

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Currying Revisited II

— How is a curried function defined? —

- A curried function of n arguments (of type $\mathtt{t}_1, \mathtt{t}_2, \cdots, \mathtt{t}_n$) that returns a value of type `t` is defined like this:

$$\mathtt{fun} \;::\; \mathtt{t}_1 \;\texttt{->}\; \mathtt{t}_2 \;\texttt{->}\; \cdots \;\texttt{->}\; \mathtt{t}_n \;\texttt{->}\; \mathtt{t}$$

- This is sort of like defining n different functions (one for each `->`). In fact, we could define these functions explicitly, but that would be tedious:

$$\mathtt{fun}_1 \;::\; \mathtt{t}_2 \;\texttt{->}\; \cdots \;\texttt{->}\; \mathtt{t}_n \;\texttt{->}\; \mathtt{t}$$
$$\mathtt{fun}_1 \; \mathtt{a}_2 \cdots \mathtt{a}_n \;=\; \cdots$$

$$\mathtt{fun}_2 \;::\; \mathtt{t}_3 \;\texttt{->}\; \cdots \;\texttt{->}\; \mathtt{t}_n \;\texttt{->}\; \mathtt{t}$$
$$\mathtt{fun}_2 \; \mathtt{a}_3 \cdots \mathtt{a}_n \;=\; \cdots$$

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Currying Revisited III

Duh, how about an example?

- Certainly. Lets define a recursive function `get_nth n xs` which returns the `n`:th element from the list `xs`:

```
get_nth 1 (x:_) = x
get_nth n (_:xs) = get_nth (n-1) xs

get_nth 10 "Bartholomew" ⇒ 'e'
```

- Now, let's use `get_nth` to define functions `get_second`, `get_third`, `get_fourth`, and `get_fifth`, without using explicit recursion:

```
get_second = get_nth 2
get_third = get_nth 3
get_fourth = get_nth 4
get_fifth = get_nth 5
```

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Currying Revisited IV

```
get_fifth "Bartholomew" ⇒ 'h'

map (get_nth 3)
   ["mob","sea","tar","bat"] ⇒
   "bart"
```

So, what's the type of `get_second`?

- Remember the **Rule of Cancellation**?
- The type of `get_nth` is `Int -> [a] -> a`.
- `get_second` applies `get_nth` to one argument. So, to get the type of `get_second` we need to cancel `get_nth`'s first type: `~~Int~~ -> [a] -> a` ≡ `[a] -> a`.

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Patterns of Computation

Mappings

- Apply a function f to the elements of a list L to make a new list L'. **Example:** Double the elements of an integer list.

Selections

- Extract those elements from a list L that satisfy a predicate p into a new list L'. **Example:** Extract the even elements from an integer list.

Folds

- Combine the elements of a list L into a single element using a binary function f. **Example:** Sum up the elements in an integer list.

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The `map` Function I

- `map` takes two arguments, a function and a list. `map` creates a new list by applying the function to each element of the input list.
- `map`'s first argument is a function of type `a -> b`. The second argument is a list of type `[a]`. The result is a list of type `[b]`.

```
map ::  (a -> b) -> [a] -> [b]
map f [ ]          = [ ]
map f (x:xs)       = f x :  map f xs
```

- We can check the type of an object using the `:type` command. Example: `:type map`.

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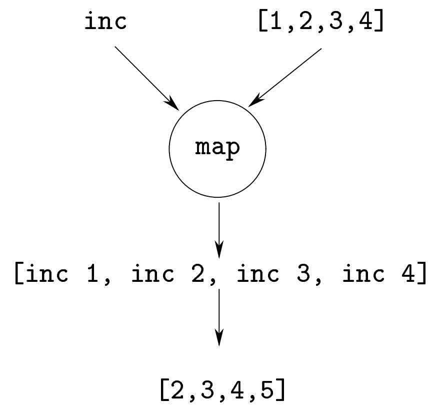
The map Function II

```
map :: (a -> b) -> [a] -> [b]
map f [ ]        = [ ]
map f (x:xs)     = f x : map f xs

inc x = x + 1
```

`map inc [1,2,3,4]` ⇒ `[2,3,4,5]`



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The map Function III

```
map :: (a -> b) -> [a] -> [b]
map f [ ]        = [ ]
map f (x:xs)     = f x : map f xs
```

map f [ ] = [ ] means: "The result of applying the function `f` to the elements of an empty list is the empty list."

map f (x:xs) = f x : map f xs means: "applying `f` to the list `(x:xs)` is the same as applying `f` to `x` (the first element of the list), then applying `f` to the list `xs`, and then combining the results."

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The map Function IV

Simulation:

```
map square [5,6] ⇒
  square 5 : map square [6] ⇒
  25 : map square [6] ⇒
    25 : (square 6 : map square [ ]) ⇒
    25 : (36 : map square [ ]) ⇒
      25 : (36 : [ ]) ⇒
        25 : [36] ⇒
          [25,36]
```

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The filter Function I

- Filter takes a predicate p and a list L as arguments. It returns a list L' consisting of those elements from L that satisfy p.
- The predicate p should have the type `a -> Bool`, where `a` is the type of the list elements.

Examples:

```
filter even [1..10] ⇒ [2,4,6,8,10]
filter even (map square [2..5]) ⇒
  filter even [4,9,16,25] ⇒
  [4,16]
filter gt10 [2,5,9,11,23,114]
  where gt10 x = x > 10 ⇒
  [11,23,114]
```

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The filter Function II

- We can define `filter` using either recursion or list comprehension.

Using recursion:

```
filter :: (a -> Bool) -> [a] -> [a]
filter _ [] = []
filter p (x:xs)
   | p x       = x : filter p xs
   | otherwise = filter p xs
```

Using list comprehension:

```
filter :: (a -> Bool) -> [a] -> [a]
filter p xs = [x | x <- xs, p x]
```

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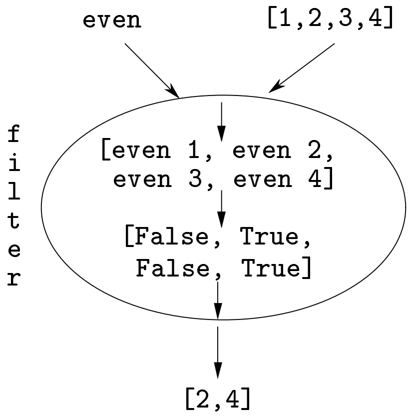
The filter Function III

```
filter :: (a -> Bool) -> [a] -> [a]
filter _ [] = []
filter p (x:xs)
   | p x       = x : filter p xs
   | otherwise = filter p xs
```

`filter even [1,2,3,4]` ⇒ `[2,4]`



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The filter Function IV

filter examples:

- `doublePos` doubles the positive integers in a list.

```
getEven :: [Int] -> [Int]
getEven xs = filter even xs

doublePos :: [Int] -> [Int]
doublePos xs
   = map dbl (filter pos xs)
      where dbl x = 2 * x
            pos x = (0 < x)
```

Simulations:

```
getEven [1,2,3] ⇒ [2]

doublePos [1,2,3,4] ⇒
   map dbl (filter pos [1,2,3,4]) ⇒
   map dbl [2,4] ⇒ [4,8]
```

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fold Functions I

- A common operation is to combine the elements of a list into one element. Such operations are called **reductions** or **accumulations**.

Examples:

```
sum [1,2,3,4,5] ≡
   (1 + (2 + (3 + (4 + (5 + 0))))) ⇒
     15
concat ["H","i","!"] ≡
   ("H" ++ ("i" ++ ("!" ++ ""))) ⇒
     "Hi!"
```

- Notice how similar these operations are. They both combine the elements in a list using some binary operator (`+`, `++`), starting out with a "seed" value (`0`, `""`).

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fold Functions II

- Gofer provides a function `foldr` ("fold right") which captures this pattern of computation.
- `foldr` takes three arguments: a function, a seed value, and a list.

Examples:

```
foldr (+) 0 [1,2,3,4,5] ⇒ 15
foldr (++) "" ["H","i","!"] ⇒ "Hi!"
```

foldr:

```
foldr :: (a->b->b) -> b -> [a] -> b
foldr f z [ ]     = z
foldr f z (x:xs)  = f x (foldr f z xs)
```

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fold Functions III

- Note how the fold process is started by combining the last element $\mathtt{x}_n$ with `z`. Hence the name **seed**.

$$\mathtt{foldr}(\oplus)\mathtt{z}[\mathtt{x}_1 \cdots \mathtt{x}_n] = (\mathtt{x}_1 \oplus (\mathtt{x}_2 \oplus (\cdots (\mathtt{x}_n \oplus \mathtt{z}))))$$

- Several functions in the standard prelude are defined using `foldr`:

```
and,or :: [Bool] -> Bool
and xs = foldr (&&) True xs
or xs = foldr (||) False xs
```

```
? or [True,False,False] ⇒
  foldr (||) False [True,False,False] ⇒
  True || (False || (False || False)) ⇒
  True
```

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fold Functions IV

- Remember that `foldr` binds from the right:

```
foldr (+) 0 [1,2,3] ⇒ (1+(2+(3+0)))
```

- There is another function `foldl` that binds from the left:

```
foldl (+) 0 [1,2,3] ⇒ (((0+1)+2)+3)
```

- In general:

$$\mathtt{foldl}(\oplus)\mathtt{z}[\mathtt{x}_1 \cdots \mathtt{x}_n] = (((\mathtt{z} \oplus \mathtt{x}_1) \oplus \mathtt{x}_2) \oplus \cdots \oplus \mathtt{x}_n)$$

- In the case of `(+)` and many other functions

$$\mathtt{foldl}(\oplus)\mathtt{z}[\mathtt{x}_1 \cdots \mathtt{x}_n] \;=\; \mathtt{foldr}(\oplus)\mathtt{z}[\mathtt{x}_1 \cdots \mathtt{x}_n]$$

- However, one version may be more efficient than the other.

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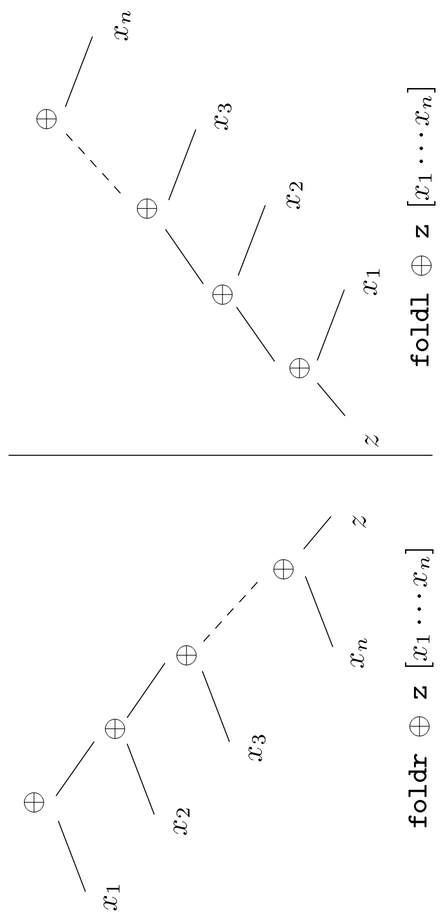
fold Functions V



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Operator Sections I

- We've already seen that it is possible to use operators to construct new functions:

(*2) – function that doubles its argument

(>2) – function that returns True for numbers > 2.

- Such **partially applied operators** are know as **operator sections**. There are two kinds:

```
(op a) b = b op a
```

```
(*2) 4           = 4 * 2  = 8
(>2) 4           = 4 > 2  = True
(++ "\n") "Bart"  = "Bart" ++ "\n"
```

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Operator Sections II

```
(a op) b = a op b
```

```
(3:)  [1,2] = 3 :  [1,2]= [3,1,2]
(0<) 5      = 0 < 5     = True
(1/)        = 1/5
```

Examples:

(+1) – The successor function.

(/2) – The halving function.

(:[]) – The function that turns an element into a singleton list.

More Examples:

```
? filter (0<) (map (+1) [-2,-1,0,1])
  [-1]
```

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takeWhile & dropWhile I

- We've looked at the **list-breaking** functions drop & take:

```
take 2 ['a','b','c'] ⇒ ['a','b']
drop 2 ['a','b','c'] ⇒ ['c']
```

- takeWhile and dropWhile are higher-order list-breaking functions. They take/drop elements from a list while a predicate is true.

```
takeWhile even [2,4,6,5,7,4,1] ⇒
  [2,4,6]
dropWhile even [2,4,6,5,7,4,1] ⇒
  [5,7,4,1]
```

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takeWhile & dropWhile II

```
takeWhile ::  (a->Bool) -> [a] -> [a]
takeWhile p [ ] = [ ]
takeWhile p (x:xs)
   | p x       = x :  takeWhile p xs
   | otherwise = [ ]

dropWhile ::  (a->Bool) -> [a] -> [a]
dropWhile p [ ] = [ ]
dropWhile p (x:xs)
   | p x       = dropWhile p xs
   | otherwise = x:xs
```

- Remove initial/final blanks from a string:

```
dropWhile ((==) '␣') "␣␣␣Hi!" ⇒
  "Hi!"
takeWhile ((/=) '␣') "Hi!␣␣␣" ⇒
  "Hi!"
```

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Summary I

- B&W reading list:
  1. Currying: 12–13
  2. Map & filter: 61–65
  3. Fold: 65–71
  4. Operator sections: 19–20
  5. Higher-order functions: 37–38
- Higher-order functions take functions as arguments, or return a function as the result.
- We can form a new function by applying a curried function to some (but not all) of its arguments. This is called **partial application**.
- **Operator sections** are partially applied infix operators.

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Summary II

- The standard prelude contains many useful higher-order functions:

  map f xs creates a new list by applying the function `f` to every element of a list `xs`.

  filter p xs creates a new list by selecting only those elements from `xs` that satisfy the predicate `p` (i.e. `(p x)` should return `True`).

  foldr f z xs reduces a list `xs` down to one element, by applying the binary function `f` to successive elements, starting from the right.

  scanl/scanr f z xs perform the same functions as `foldr/foldl`, but instead of returning only the ultimate value they return a list of all intermediate results.

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Homework I

Homework I (a):

- Define the `map` function using a list comprehension.

Template:

```
map f x = [ ··· | ··· ]
```

Homework I (b):

- Use `map` to define a function `lengthall xss` which takes a list of strings `xss` as argument and returns a list of their lengths as result.

Examples:

```
?  lengthall ["Ay", "Caramba!"]
   [2,8]
```

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Homework II

1. Give a accumulative recursive definition of `foldl`.
2. Define the `minimum xs` function using `foldr`.
3. Define a function `sumsq n` that returns the sum of the squares of the numbers $[1 \cdots n]$. Use `map` and `foldr`.
4. What does the function `mystery` below do?

```
mystery xs =
   foldr (++) [] (map sing xs)
sing x = [x]
```

Examples:

```
minimum [3,4,1,5,6,3] ⇒ 1
```

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Homework III

- Define a function `zipp f xs ys` that takes a function `f` and two lists `xs=[`$x_1, \cdots, x_n$`]` and `ys=[`$y_1, \cdots, y_n$`]` as argument, and returns the list `[f `x_1` `$y_1, \cdots,$` f `x_n` `y_n`]` as result.
- If the lists are of unequal length, an error should be returned.

Examples:

```
zipp (+) [1,2,3] [4,5,6]
  ⇒ [5,7,9]

zipp (==) [1,2,3] [4,2,2]
  ⇒ [False,True,True]

zipp (==) [1,2,3] [4,2]
  ⇒ ERROR
```

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Homework IV

- Define a function `filterFirst p xs` that removes the first element of `xs` that does not have the property `p`.

Example:

```
filterFirst even [2,4,6,5,6,8,7] ⇒
  [2,4,6,6,8,7]
```

- Use `filterFirst` to define a function `filterLast p xs` that removes the last occurence of an element of `xs` without the property `p`.

Example:

```
filterLast even [2,4,6,5,6,8,7] ⇒
  [2,4,6,5,6,8]
```

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