History of Public Key Cryptography

- RSA Conference 2011-Opening-Giants Among Us:
  http://www.youtube.com/watch?v=mvOsb9vNIWM&feature=related

- Rivest, Shamir, Adleman - The RSA Algorithm Explained:
  http://www.youtube.com/watch?v=b57zGAKKnKc

- Bruce Schneier - Who are Alice & Bob?:
  http://www.youtube.com/watch?v=BuUSi_QvFLY&feature=related

- Bruce Schneier facts:  http://www.schneierfacts.com

- Adventures of Alice & Bob - Alice Gets Lost:
  http://www.youtube.com/watch?v=nULAC_g22So  http://www.youtube.com/watch?v=nJB7a79ahGM
Public-key Algorithms

Definition (Public-key Algorithms)

Public-key cryptographic algorithms use different keys for encryption and decryption.

- Bob’s public key: $P_B$
- Bob’s secret key: $S_B$

\[
E_{P_B}(M) = C \\
D_{S_B}(C) = M \\
D_{S_B}(E_{P_B}(M)) = M
\]
Public Key Protocol

- Key-management is the main problem with symmetric algorithms – Bob and Alice have to somehow agree on a key to use.
- In public key cryptosystems there are two keys, a public one used for encryption and and private one for decryption.
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1. Alice and Bob agree on a public key cryptosystem.
2. Bob sends Alice his public key, or Alice gets it from a public database.
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3. Alice encrypts her plaintext using Bob’s public key and sends it to Bob.
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3. Alice encrypts her plaintext using Bob’s public key and sends it to Bob.
4. Bob decrypts the message using his private key.
Public Key Encryption Protocol...
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Public Key Encryption Protocol...
Public Key Encryption Protocol...
Public Key Encryption Protocol. . .

Alice

plaintext

encrypt

Bob

ciphertext

decrypt

plaintext

$P_B$

Eve

$S_B$
Public Key Encryption: Key Distribution

Advantages: $n$ key pairs to communicate between $n$ parties.

Disadvantages: Ciphers (RSA, ...) are slow; keys are large.
A Hybrid Protocol

- In practice, public key cryptosystems are not used to encrypt messages – they are simply too slow.
- Instead, public key cryptosystems are used to encrypt keys for symmetric cryptosystems. These are called session keys, and are discarded once the communication session is over.
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2. Alice generates a session key $K$, encrypts it with Bob’s public key, and sends it to Bob.
3. Bob decrypts the message using his private key to get the session key $K$. 
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1. Bob sends Alice his public key.
2. Alice generates a session key $K$, encrypts it with Bob’s public key, and sends it to Bob.
3. Bob decrypts the message using his private key to get the session key $K$.
4. Both Alice and Bob communicate by encrypting their messages using $K$. 
Hybrid Encryption Protocol...
Hybrid Encryption Protocol...
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Hybrid Encryption Protocol...

Alice

\[ K \rightarrow \text{encrypt} \rightarrow E_{P_B}(K) \rightarrow \text{decrypt} \rightarrow K \]

Bob

\[ M \rightarrow \text{encrypt} \rightarrow E_K(M) \rightarrow \text{decrypt} \rightarrow M \]
Outline

1. Introduction
2. RSA
   - Algorithm
   - Example
   - Correctness
   - Security
   - Problems
3. GPG
4. Elgamal
   - Discrete Logarithms
   - Algorithm
   - Example
   - Correctness
   - Security
   - Problems
5. Diffie-Hellman Key Exchange
   - Diffie-Hellman Key Exchange
   - Example
   - Correctness
   - Security
5. Summary
RSA

- RSA is the best known public-key cryptosystem. Its security is based on the (believed) difficulty of factoring large numbers.
- Plaintexts and ciphertexts are large numbers (1000s of bits).
- Encryption and decryption is done using modular exponentiation.
RSA: Algorithm

- **Bob** (Key generation):
RSA: Algorithm

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- **Bob (decrypt a message \( C \) received from Alice):**
  1. Compute \( M = C^d \mod n \).
How should we choose \( e \)?
- It doesn’t matter for security; everybody could use the same \( e \).
- It matters for performance: 3, 17, or 65537 are good choices.

\( n \) is referred to as the **modulus**, since it’s the \( n \) of \( \mod n \).

You can only encrypt messages \( M < n \). Thus, to encrypt larger messages you need to break them into pieces, each \( < n \).

Throw away \( p \), \( q \), and \( \phi(n) \) after the key generation stage.

Encrypting and decrypting requires a single modular exponentiation.
Select two primes: $p = 47$ and $q = 71$. 

RSA Example: Key Generations
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4. Select \( e = 79 \).
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\[
d = e^{-1} \mod \phi(n)
= 79^{-1} \mod 3220
= 1019
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6. $P = (79, 3337)$ is the RSA public key.
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6. $P = (79, 3337)$ is the RSA public key.
7. $S = (1019, 3337)$ is the RSA private key.
RSA Example: Encryption

Encrypt $M = 6882326879666683$. 
RSA Example: Encryption

1. Encrypt $M = 6882326879666683$.
2. Break up $M$ into 3-digit blocks:

$$m = \langle 688, 232, 687, 966, 668, 003 \rangle$$

Note the padding at the end.
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   Note the padding at the end.

3. Encrypt each block:
   
   $c_1 = m_1^e \mod n$
   
   $= 688^{79} \mod 3337$
   
   $= 1570$

   We get:
   
   $c = \langle 1570, 2756, 2091, 2276, 2423, 158 \rangle$
RSA Example: Decryption

Decrypt each block:

\[
\begin{align*}
m_1 &= c_1^d \mod n \\
    &= 1570^{1019} \mod 3337 \\
    &= 688
\end{align*}
\]
In-Class Exercise: Goodrich & Tamassia R-8.18

Show the result of encrypting $M = 4$ using the public key $(e, n) = (3, 77)$ in the RSA cryptosystem.
Alice is telling Bob that he should use a pair of the form

$$(3, n)$$

or

$$(16385, n)$$

as his RSA public key if he wants people to encrypt messages for him from their cell phones.

As usual, $n = pq$, for two large primes, $p$ and $q$.

What is the justification for Alice’s advice?
In-Class Exercise: Stallings pp. 270-271

1. Generate an RSA key-pair using \( p = 17, \; q = 11, \; e = 7 \).
2. Encrypt \( M = 88 \).
3. Decrypt the result from 2.
RSA Correctness

We have

\[ C = M^e \mod n \]
\[ M = C^d \mod n. \]
RSA Correctness

- We have

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- To show correctness we have to show that decryption of the ciphertext actually gets the plaintext back, i.e that, for all \( M < n \)

\[ C^d \mod n = (M^e)^d \mod n \]
\[ = M^{ed} \mod n \]
\[ = M \]
RSA Correctness: Case 1

- From the key generation step we have

\[ d = e^{-1} \mod \phi(n) \]

d from which we can conclude that

\[ ed \mod \phi(n) = 1 \]

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  \[ = M \cdot (M^{\phi(n)})^k \mod n \]
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RSA Correctness: Case 1…

- $M^{\phi(n)} \mod n = 1$ follows from Euler’s theorem.

Theorem (Euler)

Let $x$ be any positive integer that’s relatively prime to the integer $n > 0$, then

$x^{\phi(n)} \mod n = 1$
RSA Correctness: Case 2

Assume that $M$ is not relatively prime to $n$, i.e. $M$ has some factor in common with $n$, since $M < n$. 


RSA Correctness: Case 2

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  - $M$ is relatively prime with $q$ and $M = ip$, or
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- We consider only the first case, the second is similar.
RSA Correctness: Case 2.

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By Euler’s theorem we have that

$$M^{k\phi(n)} \mod q = M^{k\phi(p)\phi(q)} \mod q$$

$$= (M^{k\phi(p)})^{\phi(q)} \mod q$$

$$= 1$$
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- Thus, for some integer \( h \)
  \[ M^{k\phi(n)} = 1 + hq \]

- Multiply both sides by \( M \)
  \[
  M \cdot M^{k\phi(n)} = M(1 + hq) \\
  M^{k\phi(n)+1} = M + Mhq
  \]
RSA Correctness: Case 2. . .

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RSA Security

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If she knew \( \phi(n) = (p - 1)(q - 1) \) she could compute \( d = e^{-1} \mod \phi(n) \) using Euclid’s algorithm.

If she could factor \( n \), she’d get \( p \) and \( q \)!
Security of Cryptosystems by Failed Cryptanalysis

Propose a cryptographic scheme.
Propose a cryptographic scheme.
If an attack is found, patch the scheme. GOTO 2.
Security of Cryptosystems by Failed Cryptanalysis

1. Propose a cryptographic scheme.
2. If an attack is found, patch the scheme. GOTO 2.
3. If enough time has passed $\Rightarrow$ The scheme is secure!
Security of Cryptosystems by Failed Cryptanalysis

1. Propose a cryptographic scheme.
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3. If enough time has passed ⇒ The scheme is secure!
Security of Cryptosystems by Failed Cryptanalysis

1. Propose a cryptographic scheme.
2. If an attack is found, patch the scheme. GOTO 2.
3. If enough time has passed ⇒ The scheme is secure!

- How long is enough?
  1. It took 5 years to break the Merkle-Hellman cryptosystem.
  2. It took 10 years to break the Chor-Rivest cryptosystem.
RSA Security...

- If we can factor $n$, we can find $p$ and $q$ and the scheme is broken.
RSA Security.

- If we can factor \( n \), we can find \( p \) and \( q \) and the scheme is broken.
- As far as we know, factoring is hard.
RSA Security...

- If we can factor $n$, we can find $p$ and $q$ and the scheme is broken.
- As far as we know, factoring is hard.
- We need $n$ to be large enough, 2,048 bits.
RSA Factoring Challenge

On December 3, 2003, a team of researchers in Germany and several other countries reported a successful factorization of the challenge number RSA-576.

The factors are:

\[
\begin{align*}
398075086424064937397125500550386491199064362 \\
342526708406385189575946388957261768583317 \\
472772146107435302536223071973048224632914695 \\
302097116459852171130520711256363590397527 
\end{align*}
\]
The factoring research team of F. Bahr, M. Boehm, J. Franke, T. Kleinjung continued its productivity with a successful factorization of the challenge number RSA-640, reported on November 2, 2005.

The factors are:

\[ \begin{align*}
16347336458092538484431338838650908598417836700330 \\
92312181110852389333100104508151212118167511579 \\
1900871281664822113126851573935413975471896789968 \\
515493666638539088027103802104498957191261465571
\end{align*} \]

The effort took approximately 30 2.2GHz-Opteron-CPU years according to the submitters, over five months of calendar time.
## RSA Factoring Challenge

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RSA Factoring Challenge...

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<tr>
<td></td>
<td>040445364023527381951378636564391212010397122822120720357</td>
</tr>
</tbody>
</table>
RSA Security: How to use RSA

- Two plaintexts $M_1$ and $M_2$ are encrypted into ciphertexts $C_1$ and $C_2$. 
RSA Security: How to use RSA

- Two plaintexts $M_1$ and $M_2$ are encrypted into ciphertexts $C_1$ and $C_2$.
- But, RSA is deterministic!
RSA Security: How to use RSA

- Two plaintexts $M_1$ and $M_2$ are encrypted into ciphertexts $C_1$ and $C_2$.
- But, RSA is deterministic!
- If $C_1 = C_2$ then we know that $M_1 = M_2$!
RSA Security: How to use RSA

- Two plaintexts $M_1$ and $M_2$ are encrypted into ciphertexts $C_1$ and $C_2$.
- But, RSA is deterministic!
- If $C_1 = C_2$ then we know that $M_1 = M_2$!
- Also, side-channel attacks are possible against RSA, for example by measuring the time taken to encrypt.
In-class Exercise: 2012 Midterm Exam

- Generate an RSA key-pair using $p = 11$, $q = 13$, $e = 7$. Show your work!
In-class Exercise: 2012 Midterm Exam

Given the RSA public key $P = (7, 65)$ and secret key $S = (29, 65)$, encrypt $M = 5$. Make sure to use an efficient method of computation. Show your work!
Outline

1. Introduction
2. RSA
   - Algorithm
   - Example
   - Correctness
   - Security
   - Problems
3. GPG
4. Elgamal
   - Discrete Logarithms
   - Algorithm
   - Example
   - Correctness
   - Security
   - Problems
5. Diffie-Hellman Key Exchange
   - Diffie-Hellman Key Exchange
   - Example
   - Correctness
   - Security
6. Summary
Software – GPG

- gpg is a public domain implementation of pgp.
- Supported algorithms:
  - Pubkey: RSA, RSA-E, RSA-S, ELG-E, DSA
  - Cipher: 3DES, CAST5, BLOWFISH, AES, AES192, AES256, TWOFISH, CAMELLIA128, CAMELLIA192, CAMELLIA256
  - Hash: MD5, SHA1, RIPEMD160, SHA256, SHA384, SHA512, SHA224
  - Compression: Uncompressed, ZIP, ZLIB, BZIP2

- [http://www.gnupg.org](http://www.gnupg.org)
Key generation: Bob

> gpg --gen-key

Please select what kind of key you want:
   (1) RSA and RSA (default)
   (2) DSA and Elgamal
   (3) DSA (sign only)
   (4) RSA (sign only)
Your selection? 1
What keysize do you want? (2048)
Key is valid for? (0)
Key does not expire at all
Real name: Bobby
Email address: bobby@gmail.com
Comment: recipient
You need a Passphrase to protect your secret key.
Enter passphrase: Bob rocks
Repeat passphrase: Bob rocks
Key generation: Alice

> gpg --gen-key

Please select what kind of key you want:
(1) RSA and RSA (default)
(2) DSA and Elgamal
(3) DSA (sign only)
(4) RSA (sign only)
Your selection? 1
What keysize do you want? (2048)
Key is valid for? (0)
Key does not expire at all
Real name: Alice
Email address: alice@gmail.com
Comment: sender
You need a Passphrase to protect your secret key.
Enter passphrase: Alice is cute
Repeat passphrase: Alice is cute
Exporting the Key

> gpg --armor --export Bobby
-----BEGIN GPG PUBLIC KEY BLOCK-----
Version: GnuPG v1.4.11 (Darwin)

mQENBE83U28BCADTV0kHpNjWzk7yEzMhiNJcm0tmUYfn4hzgYTDsP2otI0UhfJ4qEZCuPoxECIZ479k3YpBvZM2JC48Ht9j1kVnDPLCrongyRdSko0AwG70YAyHWa7/USeGwjZ+0MUuM3SwqHdo1/0XS3P8LABTQNXtrQf9kF8UNLIAhr1IvBcae1K44MPL6
............................................................
EBHmAM7iiWgW16/6qEmN46ZQEmoR86vWhQL3LQ6p/FUaBA==
=FZ78
-----END GPG PUBLIC KEY BLOCK-----
Encryption

- We can encrypt a message using Bobby’s key:

```
> cat message
Attack at dawn
> gpg --recipient bobby --armor --encrypt message
> cat message.asc
-----BEGIN PGP MESSAGE-----
Version: GnuPG v1.4.11 (Darwin)

hQEMA97v9lbZUpHvAQf/a9QklXMiMzBWy5yyZBtNrg7FcrIqx+gXVVUXNN86tZtERF42elwU6QwamDzfcOHqp+3zeor4Y5xN+/pL91xti6uwF0hgGrCGJq//AfUKgQykMH2e4gR8Y1BuPm9b1c7uzXxRMM0UBBt75KquYG0BLybsP29ttD9iL/ZJ11zSPjSjE1700Gp7PqEBotStV0tuknYW/fX0zXndU8XNllKnsnZn21Xm0rMQcFMu8Do/tF5IL1RfTEcL4S9tV4vshgXhNSpTg9sZs1UZynvU2cJqyYkCtgT7TdtrK3fTa8UN+CYQvU2QRnaNtFhYwBMonFqhefNzDqeZb+P0Rq0uoD11YuNJRAViJ3CLjT7kwgBgRtNfYRkGArQQmgrknW2jq/Y2GZTE8CC7pNXy8U3KYM19hRA6U5fMp08ndFp8vowBbB2swzjxjSY7ZeIR2uwxdLYydtW4m
=B+JA
-----END PGP MESSAGE-----
```
Decryption

- Bobby can now decrypt the message using his private key:

  > gpg --decrypt message.asc

You need a passphrase to unlock the secret key for
user: "Bobby (recipient) <bobby@gmail.com>"
2048-bit RSA key, ID D95291EF, created 2012-02-12
(main key ID 9974031B)

Enter passphrase: Bob rocks

  gpg: encrypted with 2048-bit RSA key, ID D95291EF, created 2012-02-12
  "Bobby (recipient) <bobby@gmail.com>"
Attack at dawn
The keyring

> gpg --list-keys
/Users/collberg/.gnupg/pubring.gpg

----------------------------------
pub  2048R/9974031B  2012-02-12
uid  Bobby (recipient) <bobby@gmail.com>
sub  2048R/D95291EF  2012-02-12

pub  2048R/4EC8A0CB  2012-02-12
uid  Alice (sender) <alice@gmail.com>
sub  2048R/B901E082  2012-02-12
The keyring...

```bash
> gpg --list-secret-keys
/Users/collberg/.gnupg/secring.gpg

----------------------------------
sec  2048R/9974031B  2012-02-12
uid  Bobby (recipient) <bobby@gmail.com>
ssb  2048R/D95291EF  2012-02-12

sec  2048R/4EC8A0CB  2012-02-12
uid  Alice (sender) <alice@gmail.com>
ssb  2048R/B901E082  2012-02-12
```
Sign and Encrypt

- Bob can sign his message before sending it to Alice:

  > gpg -se --recipient alice --armor message

You need a passphrase to unlock the secret key for user: "Bobby (recipient) <bobby@gmail.com>"
2048-bit RSA key, ID 9974031B, created 2012-02-12

Enter passphrase: Bob rocks

> cat message.asc

-----BEGIN PGP MESSAGE-----
Version: GnuPG v1.4.11 (Darwin)

hQEMA7osp1S5AeCCAQgAsSs+Urf0f3KHTtP7cqTwugpcJ9oUAGkw/KQ0DHIE0v
8XEAaCwZ8aZK1lXhqBSd/9hCm9Mup2NECiU8crVyff7NTWFyaTBeGA9m10q3y46o
QpIgPbcdYZqIt8e/8wPU6x1MZUStzxBKLB+Rj/Zg35ZVioYL
=oiv8
-----END PGP MESSAGE-----
Check Signature and Decrypt

- Alice can now decrypt the message and check the signature:

> gpg --decrypt message.asc

You need a passphrase to unlock the secret key for user: "Alice (sender) <alice@gmail.com>"
2048-bit RSA key, ID B901E082,
created 2012-02-12 (main key ID 4EC8A0CB)

Enter passphrase: Alice is cute

gpg: encrypted with 2048-bit RSA key, ID B901E082, created 2012-02-12
   "Alice (sender) <alice@gmail.com>"
Attack at dawn

gpg: Signature made Sat Feb 11 23:10:59 2012 MST
using RSA key ID 9974031B

gpg: Good signature from "Bobby (recipient) <bobby@gmail.com>"
Symmetric Encryption Only

```
> gpg --cipher-algo=AES --armor --symmetric message
Enter passphrase: sultana
Repeat passphrase: sultana
> cat message.asc
-----BEGIN PGP MESSAGE-----
Version: GnuPG v1.4.11 (Darwin)

jA0EByMCgZ3PBofSZxJlg0ksBBooTMLEVQ2q9HkTR5y9FIoX9nbsyohr0XeQLFLcf
wtWcg+dZv1MS6D70E3wZCeW2LX50kycU17MUc8wnJLDAzAdRqPAgDma+sP4=   
=UtI4
-----END PGP MESSAGE-----

> gpg message.asc
gpg: AES encrypted data
Enter passphrase: sultana
```
```
gpg: encrypted with 1 passphrase

> cat message
Attack at dawn
```
Deleting Keys

> gpg --delete-secret-keys bobby
sec 2048R/9974031B 2012-02-12 Bobby (recipient) <bobby@gmail.com>

Delete this key from the keyring? (y/N) y
This is a secret key! - really delete? (y/N) y

> gpg --delete-keys bobby
pub 2048R/9974031B 2012-02-12 Bobby (recipient) <bobby@gmail.com>

Delete this key from the keyring? (y/N) y
Generating Primes

- Generate a prime number of the given number of bits:

```plaintext
> gpg --gen-prime 1 16
C4B7
> gpg --gen-prime 1 1024
D34D4347ED013242EE06811BC561C6587D75ADE33D1BEC954D648E22
9D88B5E0AF1394459FB48B135B99C8BA8C50E5331C6226CBF6D70031
4A8CC84C7B363BE7DD7BBBB29E545D199339263F5FB2E9F1B84BA9D5
05B5B79858FC6149CF09E6C56D9730C3BD1E62B378C8DFAF4233B8DC
BA999A21EC9C4BF8C60AACDCBC607AC5
```
Generating Random Numbers

- Generate 100 (base64 encoded) random bytes:

```bash
> gpg --armour --gen-random 0 100
e0zAVl6jbe/Dma9VF20lMgZxE1RA4S8TwNwu6KP8+o1kjdtBm2
AjKFSVsj/d3zG/9KqmNj7j6symEUZ3e0fWzaWqLBxzJuSur5sK
C8omfPus2QtYJJNOgVbpJ7X9L4t1iNJtnw==
```
Print Message Digests

> gpg --print-mds message

MD5      = 36 D1 A5 12 17 CD 34 FC 04 F5 6C C4 91 39 C7 59
SHA1     = 6DA4 473A 00CE 7AB6 7B6F 884D 1E75 6633 C21A 56DB
RMD160   = D1DE 4194 C0CD 3AED 30F3 38CD 68F3 800F CCF0 3B87
SHA224   = B4E94780 1AA1A9C3 418F72D8 651BA995 83284003
          EBEE183A 589702EE
SHA256   = B83EF405 07696578 9D4BBDA7 D7932700 5F2AE6CB
          A2696FDE 69694D12 AFE70E4A
SHA384   = 7AC39A0C 945844F1 1316BB46 C9FC7EEA E892A178
          2D20E4CA E7BE686C 1A091C8C F1BBDFD1 3D42BEA2
          88AF5A4F E3705474
SHA512   = 9CA1EB88 F064CB0D 536254B2 5755919F 45564276
          96CA27A0 389E4817 53F81DC2 3222488D 7D11F3DD
          C066B9E8 027F3870 395A2561 157DDC38 BD679D37
          C2E361CC
Goal: Read a message encrypted with gpg

1. Decrypt the message itself (OR)

http://www.schneier.com/paper-attacktrees-fig7.html
Goal: Read a message encrypted with gpg

1. Decrypt the message itself (OR)
2. Determine symmetric key used to encrypt the message by other means (OR)

http://www.schneier.com/paper-attacktrees-fig7.html
Goal: Read a message encrypted with gpg

1. Decrypt the message itself (OR)
2. Determine symmetric key used to encrypt the message by other means (OR)
3. Get recipient to help decrypt message (OR)

http://www.schneier.com/paper-attacktrees-fig7.html
Goal: Read a message encrypted with gpg

1. Decrypt the message itself (OR)
2. Determine symmetric key used to encrypt the message by other means (OR)
3. Get recipient to help decrypt message (OR)
4. Obtain private key of recipient.

http://www.schneier.com/paper-attacktrees-fig7.html
Goal: Read a message encrypted with gpg...

Decrypt the message itself:

1. Break asymmetric encryption (OR)
   1. Brute force break asymmetric encryption (OR)
   2. Mathematically break asymmetric encryption (OR)
      1. Break RSA (OR)
      2. Factor RSA modulus/calculate Elgamal discrete log
   3. Cryptanalyze asymmetric encryption (OR)
      1. General cryptanalysis of RSA/Elgamal (OR)
      2. Exploit weakness in RSA/Elgamal (OR)
      3. Timing attack on RSA/Elgamal

2. Break symmetric-key encryption
   1. Brute force break symmetric-key encryption
   2. Cryptanalysis of symmetric-key encryption
Goal: Read a message encrypted with gpg...

Determine symmetric key by other means:

1. Fool sender into encrypting message using public key whose private key is known (OR)
   - Convince sender that fake key (with known private key) is the key of the intended recipient
   - Convince sender to encrypt with more than one key—the real key of the recipient and a key whose private key is known.
   - Have the message encrypted with a different public key in the background, unbeknownst to the sender.
2. Have the recipient sign the encrypted public key (OR)
3. Monitor the sender’s computer memory (OR)
4. Monitor the receiver’s computer memory (OR)
5. Determine key from pseudo-random number generator (OR)
   - Determine state of randseed during encryption (OR)
   - Implant virus that alters the state of randseed. (OR)
   - Implant software that affects the choice of symmetric key.
6. Implant virus that exposes public key.
Goal: Read a message encrypted with gpg...

Get recipient to help decrypt message:
Goal: Read a message encrypted with gpg...

Obtain private key of recipient:
Goal: Read a message encrypted with PGP

What immediately becomes apparent from the attack tree is that breaking the RSA or IDEA encryption algorithms are not the most profitable attacks against PGP. There are many ways to read someone’s PGP-encrypted messages without breaking the cryptography. You can capture their screen when they decrypt and read the messages (using a Trojan horse like Back Orifice, a TEMPEST receiver, or a secret camera), grab their private key after they enter a passphrase (Back Orifice again, or a dedicated computer virus), recover their passphrase (a keyboard sniffer, TEMPEST receiver, or Back Orifice), or simply try to brute force their passphrase (I can assure you that it will have much less entropy than the 128-bit IDEA keys that it generates).
In the scheme of things, the choice of algorithm and the key length is probably the least important thing that affects PGP’s overall security. PGP not only has to be secure, but it has to be used in an environment that leverages that security without creating any new insecurities.

http://www.schneier.com/paper-attacktrees-fig7.html
Outline

1 Introduction
2 RSA
   - Algorithm
   - Example
   - Correctness
   - Security
   - Problems
3 GPG
4 Elgamal
   - Discrete Logarithms
   - Algorithm
   - Example
   - Correctness
   - Security
   - Problems
5 Diffie-Hellman Key Exchange
   - Diffie-Hellman Key Exchange
   - Example
   - Correctness
   - Security
5 Summary
The Elgamal cryptosystem relies on the inherent difficulty of calculating discrete logarithms.

It is a probabilistic scheme:
- a particular plaintext can be encrypted into multiple different ciphertexts;
- \( \Rightarrow \) ciphertexts become twice the length of the plaintext.

RSA Conference 2009 Lifetime Achievement Award: Taher Elgamal: [http://www.youtube.com/watch?v=ZuXUeBiE2r0](http://www.youtube.com/watch?v=ZuXUeBiE2r0)
Review of Discrete Logarithms

- All the powers of $a$, modulo 19.
- The length of the sequence is highlighted.

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Primitive Roots

- All sequences end with 1.
Primitive Roots

- All sequences end with 1.
- Some sequences have length 18. Then we say
Primitive Roots

- All sequences end with 1.
- Some sequences have length 18. Then we say
  - \( a \) generates the set of nonzero integers, modulo 19.
Primitive Roots

- All sequences end with 1.
- Some sequences have length 18. Then we say
  - \( a \) generates the set of nonzero integers, modulo 19.
  - \( a \) is a primitive root of the modulus 19.
Primitive Roots

- All sequences end with 1.
- Some sequences have length 18. Then we say
  - $a$ generates the set of nonzero integers, modulo 19.
  - $a$ is a primitive root of the modulus 19.
- If $a$ is a primitive root of $n$ then all its powers
  \[ a, a^2, \ldots, a^{\phi(n)} \]
  are distinct.
Primitive Roots

- All sequences end with 1.
- Some sequences have length 18. Then we say
  - \( a \) generates the set of nonzero integers, modulo 19.
  - \( a \) is a primitive root of the modulus 19.
- If \( a \) is a primitive root of \( n \) then all its powers
  \[ a, a^2, \ldots, a^{\phi(n)} \]
  are distinct.
- If \( a \) is a primitive root of \( p \), and \( p \) is prime, then
  \[ a, a^2, \ldots, a^p \]
  are distinct \( \mod p \).
For example, looking at the table above, we see that 2 is a primitive root modulo 19:

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because for each integer $i \in \mathbb{Z}_{19} = \{1, 2, 3, \ldots, 18\}$ there's an integer $k$, such that $i = 2^k \mod 19$. 
For example, looking at the table above, we see that 2 is a primitive root modulo 19:

\[
\begin{array}{cccccccccccccccc}
2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 & 2^8 & 2^9 & 2^{10} & 2^{11} & 2^{12} & 2^{13} & 2^{14} & 2^{15} & 2^{16} & 2^{17} & 2^{18} \\
2 & 4 & 8 & 16 & 13 & 7 & 14 & 9 & 18 & 17 & 15 & 11 & 3 & 6 & 12 & 5 & 10 & 1 \\
\end{array}
\]

because for each integer \( i \in \mathbb{Z}_{19} = \{1, 2, 3, \ldots, 18\} \) there’s an integer \( k \), such that \( i = 2^k \mod 19 \).

There are \( \phi(p - 1) \) generators for \( \mathbb{Z}_p \).
Computing Primitive Roots

Consider the equation

\[ y = g^x \mod p \]

If we have \( g, x, \) and \( p \) it's easy to calculate \( y \).
Computing Primitive Roots

- Consider the equation

\[ y = g^x \mod p \]

If we have \( g, x, \) and \( p \) it's easy to calculate \( y \).
- What if, instead, we're given \( y, g, \) and \( p \)?
Computing Primitive Roots

Consider the equation

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If we have \( g, x, \) and \( p \) it’s easy to calculate \( y \).

What if, instead, we’re given \( y, g, \) and \( p \)?

It’s hard to take the discrete logarithm, i.e. to compute \( x \).
Computing Primitive Roots

- Consider the equation

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If we have \( g, x, \) and \( p \) it’s easy to calculate \( y \).

- What if, instead, we’re given \( y, g, \) and \( p \)?
  - it’s hard to take the discrete logarithm, i.e. to compute \( x \).

- The fastest known algorithm is

\[ O\left(e^{\left((\ln p)^{1/3}(\ln(\ln p))^{2/3}\right)}\right) \]

which is infeasible for large primes \( p \).
Elgamal: Algorithm

- **Bob** (Key generation):
Elgamal: Algorithm

- **Bob** (Key generation):
  - Pick a prime $p$. 
Elgamal: Algorithm

1. Bob (Key generation):
   1. Pick a prime $p$.
   2. Find a generator $g$ for $\mathbb{Z}_p$. 
Elgamal: Algorithm

- **Bob** (Key generation):
  1. Pick a prime $p$.
  2. Find a generator $g$ for $\mathbb{Z}_p$.
  3. Pick a random number $x$ between 1 and $p - 2$. 
Elgamal: Algorithm

**Bob** (Key generation):

1. Pick a prime $p$.
2. Find a generator $g$ for $Z_p$.
3. Pick a random number $x$ between 1 and $p - 2$.
4. Compute $y = g^x \mod p$. 
Elgamal: Algorithm

- **Bob** (Key generation):
  1. Pick a prime \( p \).
  2. Find a generator \( g \) for \( \mathbb{Z}_p \).
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  4. Compute \( y = g^x \mod p \).

  \( P_B = (p, g, y) \) is Bob’s Elgamal public key.
Elgamal: Algorithm

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- **Alice** (encrypt and send a message $M$ to Bob):
Elgamal: Algorithm

**Bob** (Key generation):
1. Pick a prime $p$.
2. Find a generator $g$ for $Z_p$.
3. Pick a random number $x$ between 1 and $p - 2$.
4. Compute $y = g^x \mod p$.

$P_B = (p, g, y)$ is Bob’s Elgamal public key.

$S_B = x$ is Bob’s Elgamal private key.

**Alice** (encrypt and send a message $M$ to Bob):
1. Get Bob’s public key $P_B = (p, g, y)$. 

Elgamal
Elgamal: Algorithm

- **Bob** (Key generation):
  1. Pick a prime $p$.
  2. Find a generator $g$ for $\mathbb{Z}_p$.
  3. Pick a random number $x$ between 1 and $p - 2$.
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- **Alice** (encrypt and send a message $M$ to Bob):
  1. Get Bob’s public key $P_B = (p, g, y)$.
  2. Pick a random number $k$ between 1 and $p - 2$. 

Elgamal: Algorithm

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  3. Compute the ciphertext $C = (a, b)$:

     \[
     a = g^k \mod p \\
     b = My^k \mod p
     \]
Elgamal: Algorithm

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    \]

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Elgamal: Algorithm

**Bob** (Key generation):
1. Pick a prime $p$.
2. Find a generator $g$ for $\mathbb{Z}_p$.
3. Pick a random number $x$ between 1 and $p - 2$.
4. Compute $y = g^x \mod p$.

$P_B = (p, g, y)$ is Bob’s Elgamal public key.
$S_B = x$ is Bob’s Elgamal private key.

**Alice** (encrypt and send a message $M$ to Bob):
1. Get Bob’s public key $P_B = (p, g, y)$.
2. Pick a random number $k$ between 1 and $p - 2$.
3. Compute the ciphertext $C = (a, b)$:
   
   $a = g^k \mod p$
   
   $b = My^k \mod p$ 

**Bob** (decrypt a message $C = (a, b)$ received from Alice):
1. Compute $M = b(a^x)^{-1} \mod p$. 
Alice must choose a different random number $k$ for every message, or she’ll leak information.

Bob doesn’t need to know the random value $k$ to decrypt.

Each message has $p - 1$ possible different encryptions.

The division in the decryption can be avoided by use of Lagrange’s theorem:

$$M = b \cdot (a^x)^{-1} \mod p$$

$$= b \cdot a^{p-1-x} \mod p$$
Elgamal: Finding the generator

- Computing the generator is, in general, hard.
- We can make it easier by choosing a prime number with the property that we can factor \( p - 1 \).
- Then we can test that, for each prime factor \( p_i \) of \( p - 1 \):

\[
g^{(p-1)/p_i} \mod p \neq 1
\]

If \( g \) is not a generator, then one of these powers will \( \neq 1 \).
Elgamal Example: Key generation

Elgamal Example: Key generation

2. Find a generator $g = 2$ for $Z_{13}$ (see next slide).
Elgamal Example: Key generation

2. Find a generator $g = 2$ for $\mathbb{Z}_{13}$ (see next slide).
3. Pick a random number $x = 7$. 
Elgamal Example: Key generation

1. Pick a prime \( p = 13 \).
2. Find a generator \( g = 2 \) for \( \mathbb{Z}_{13} \) (see next slide).
3. Pick a random number \( x = 7 \).
4. Compute
   \[
y = g^x \mod p = 2^7 \mod 13 = 11.
   \]
Elgamal Example: Key generation

2. Find a generator $g = 2$ for $\mathbb{Z}_{13}$ (see next slide).
3. Pick a random number $x = 7$.
4. Compute
   \[ y = g^x \mod p = 2^7 \mod 13 = 11. \]
5. $P_B = (p, g, y) = (13, 2, 11)$ is Bob’s public key.
Elgamal Example: Key generation

1. Pick a prime \( p = 13 \).
2. Find a generator \( g = 2 \) for \( Z_{13} \) (see next slide).
3. Pick a random number \( x = 7 \).
4. Compute
   \[ y = g^x \mod p = 2^7 \mod 13 = 11. \]
5. \( P_B = (p, g, y) = (13, 2, 11) \) is Bob’s public key.
6. \( S_B = x = 7 \) is Bob’ private key.
2 is a primitive root modulo 13 because for each integer \( i \in \mathbb{Z}_{13} = \{1, 2, 3, \ldots, 12\} \) there's an integer \( k \), such that \( i = 2^k \mod 13 \):

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Elgamal Example: Encryption

Encrypt the plaintext message \( M = 3 \).

- Alice gets Bob’s public key \( P_B = (p, g, y) = (13, 2, 11) \).
Elgamal Example: Encryption

Encrypt the plaintext message $M = 3$.

- Alice gets Bob’s public key $P_B = (p, g, y) = (13, 2, 11)$.
- To encrypt:
Elgamal Example: Encryption

Encrypt the plaintext message $M = 3$.

- Alice gets Bob’s public key $P_B = (p, g, y) = (13, 2, 11)$.
- To encrypt:
  1. Pick a random number $k = 5$: 
Elgamal Example: Encryption

Encrypt the plaintext message $M = 3$.

- Alice gets Bob’s public key $P_B = (p, g, y) = (13, 2, 11)$.
- To encrypt:
  1. Pick a random number $k = 5$:
  2. Compute:

$$
a = g^k \mod p = 2^5 \mod 13 = 6
$$

$$
b = My^k \mod p = 3 \cdot 11^5 \mod 13 = 8
$$
Elgamal Example: Encryption

Encrypt the plaintext message $M = 3$.

- Alice gets Bob’s public key $P_B = (p, g, y) = (13, 2, 11)$.
- To encrypt:
  1. Pick a random number $k = 5$:
  2. Compute:

$$a = g^k \mod p = 2^5 \mod 13 = 6$$
$$b = My^k \mod p = 3 \cdot 11^5 \mod 13 = 8$$

- The ciphertext $C = (a, b) = (6, 8)$. 
Elgamal Example: Decryption

- Bob’s private key is $S_B = x = 7$. 
Elgamal Example: Decryption

- Bob’s private key is $S_B = x = 7$.
- Bob receives the ciphertext $C = (a, b) = (6, 8)$ from Alice.
Elgamal Example: Decryption

- Bob’s private key is $S_B = x = 7$.
- Bob receives the ciphertext $C = (a, b) = (6, 8)$ from Alice.
- Bob computes the plaintext $M$:

\[ M = b \cdot (a^x)^{-1} \mod p \]
Elgamal Example: Decryption

- Bob’s private key is \( S_B = x = 7 \).
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\[
M = b \cdot (a^x)^{-1} \mod p
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- Bob’s private key is $S_B = x = 7$.
- Bob receives the ciphertext $C = (a, b) = (6, 8)$ from Alice.
- Bob computes the plaintext $M$:

\[
M = b \cdot (a^x)^{-1} \mod p \\
= b \cdot a^{p-1-x} \mod p
\]
Elgamal Example: Decryption

- Bob’s private key is $S_B = x = 7$.
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  $$M = b \cdot (a^x)^{-1} \mod p$$
  $$= b \cdot a^{p-1-x} \mod p$$
  $$= 8 \cdot 6^{13-1-7} \mod 13$$
Elgamal Example: Decryption

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= 8 \cdot 6^5 \mod 13
\]
Elgamal Example: Decryption

- Bob’s private key is $S_B = x = 7$.
- Bob receives the ciphertext $C = (a, b) = (6, 8)$ from Alice.
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$$
M = b \cdot (a^x)^{-1} \mod p \\
= b \cdot a^{p-1-x} \mod p \\
= 8 \cdot 6^{13-1-7} \mod 13 \\
= 8 \cdot 6^5 \mod 13 \\
= 3
$$
In-Class Exercise

- Pick the prime \( p = 13 \).
- Find the generator \( g = 2 \) for \( Z_{13} \).
- Pick a random number \( x = 9 \).
- Compute
  \[
  y = g^x \mod p = 2^9 \mod 13 = 5
  \]
- \( P_B = (p, g, y) = (13, 2, 5) \) is Bob’s public key.
- \( S_B = x = 9 \) is Bob’ private key.

1. Encrypt the message \( M = 11 \) using the random number \( k = 10 \).
2. Decrypt the ciphertext from 1.
Elgamal Correctness

- Show that $M = b \cdot (a^x)^{-1} \mod p$ decrypts.
Elgamal Correctness

- Show that $M = b \cdot (a^x)^{-1} \mod p$ decrypts.
- We have that

\[
\begin{align*}
a &= g^k \mod p \\
b &= My^k \mod p \\
y &= g^x \mod p
\end{align*}
\]
Elgamal Correctness

- Show that $M = b \cdot (a^x)^{-1} \mod p$ decrypts.
- We have that

\[
\begin{align*}
    a &= g^k \mod p \\
    b &= My^k \mod p \\
    y &= g^x \mod p
\end{align*}
\]

- We get

\[
    b \cdot (a^x)^{-1} \mod p = (M y^k) \cdot ((g^k)^x)^{-1} \mod p
\]
Elgamal Correctness

- Show that $M = b \cdot (a^x)^{-1} \mod p$ decrypts.
- We have that

\[
\begin{align*}
a &= g^k \mod p \\
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- Show that $M = b \cdot (a^x)^{-1} \mod p$ decrypts.
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    a &= g^k \mod p \\
    b &= My^k \mod p \\
    y &= g^x \mod p
\end{align*}
\]

- We get

\[
\begin{align*}
    b \cdot (a^x)^{-1} \mod p &= (My^k) \cdot ((g^k)^x)^{-1} \mod p \\
    &= (My^k) \cdot (g^{kx})^{-1} \mod p
\end{align*}
\]
Elgamal Correctness

- Show that $M = b \cdot (a^x)^{-1} \mod p$ decrypts.
- We have that

  $$
  \begin{align*}
  a &= g^k \mod p \\
  b &= My^k \mod p \\
  y &= g^x \mod p
  \end{align*}
  $$

- We get

  $$
  b \cdot (a^x)^{-1} \mod p = (My^k) \cdot ((g^k)^x)^{-1} \mod p \\
  = (My^k) \cdot (g^{kx})^{-1} \mod p \\
  = (M((g^x)^k)) \cdot (g^{kx})^{-1} \mod p
  $$
Elgamal Correctness

- Show that \( M = b \cdot (a^x)^{-1} \mod p \) decrypts.
- We have that

\[
\begin{align*}
  a &= g^k \mod p \\
  b &= My^k \mod p \\
  y &= g^x \mod p
\end{align*}
\]

- We get

\[
\begin{align*}
  b \cdot (a^x)^{-1} \mod p &= (My^k) \cdot ((g^k)^x)^{-1} \mod p \\
  &= (My^k) \cdot (g^{kx})^{-1} \mod p \\
  &= (M((g^x)^k) \cdot (g^{kx})^{-1} \mod p \\
  &= Mg^{kx} \cdot (g^{kx})^{-1} \mod p
\end{align*}
\]
Show that $M = b \cdot (a^x)^{-1} \mod p$ decrypts.

We have that

\[
\begin{align*}
    a &= g^k \mod p \\
    b &= My^k \mod p \\
    y &= g^x \mod p
\end{align*}
\]

We get

\[
\begin{align*}
    b \cdot (a^x)^{-1} \mod p &= (My^k) \cdot ((g^k)^x)^{-1} \mod p \\
    &= (My^k) \cdot (g^{kx})^{-1} \mod p \\
    &= (M((g^x)^k) \cdot (g^{kx})^{-1} \mod p \\
    &= Mg^{kx} \cdot (g^{kx})^{-1} \mod p \\
    &= Mg^{kx} \cdot g^{-kx} \mod p
\end{align*}
\]
Elgamal Correctness

- Show that \( M = b \cdot (a^x)^{-1} \mod p \) decrypts.
- We have that

\[
\begin{align*}
a &= g^k \mod p \\
b &= My^k \mod p \\
y &= g^x \mod p
\end{align*}
\]

- We get

\[
b \cdot (a^x)^{-1} \mod p = (My^k) \cdot ((g^k)^x)^{-1} \mod p \\
= (My^k) \cdot (g^{kx})^{-1} \mod p \\
= (M((g^x)^k)) \cdot (g^{kx})^{-1} \mod p \\
= Mg^{kx} \cdot (g^{kx})^{-1} \mod p \\
= Mg^{kx} \cdot g^{-kx} \mod p \\
= M \mod p
\]
Elgamal Correctness

- Show that $M = b \cdot (a^x)^{-1} \mod p$ decrypts.
- We have that

  $$a = g^k \mod p$$
  $$b = My^k \mod p$$
  $$y = g^x \mod p$$

- We get

  $$b \cdot (a^x)^{-1} \mod p = (My^k) \cdot ((g^k)^x)^{-1} \mod p$$
  $$= (My^k) \cdot (g^{kx})^{-1} \mod p$$
  $$= (M((g^x)^k) \cdot (g^{kx})^{-1} \mod p$$
  $$= Mg^{kx} \cdot (g^{kx})^{-1} \mod p$$
  $$= Mg^{kx} \cdot g^{-kx} \mod p$$
  $$= M \mod p$$
  $$= M$$
Elgamal Security

- The security of the scheme depends on the hardness of solving the discrete logarithm problem.
Elgamal Security

- The security of the scheme depends on the hardness of solving the discrete logarithm problem.
- Generally believed to be hard.
Show that the Elgamal cryptosystem is homomorphic in multiplication, i.e. that for two messages $M_1$ and $M_2$, multiplying their ciphertexts is equivalent to encrypting the multiplication of their plaintexts:

$$E(M_1) \cdot E(M_2) = E(M_1 \cdot M_2).$$
Outline

1 Introduction
2 RSA
   - Algorithm
   - Example
   - Correctness
   - Security
   - Problems
3 GPG
4 Elgamal
   - Discrete Logarithms
   - Algorithm
   - Example
   - Correctness
   - Security
   - Problems
5 Diffie-Hellman Key Exchange
   - Diffie-Hellman Key Exchange
   - Example
   - Correctness
   - Security
6 Summary
Key Exchange

- A key exchange protocol (or key agreement protocol) is a way for parties to share a secret (such as a symmetric key) over an insecure channel.

- With an active adversary (who can modify messages) we can’t reliably share a secret.

- With a passive adversary (who can only eavesdrop on messages) we can share a secret.

- A passive adversary is said to be honest but curious.
Key Exchange

- 2008 Royal Institution Christmas Lectures:
  [Link](http://www.youtube.com/watch?v=U62S8SchxX4)

- How internet security works (explained with tennis balls):
  [Link](http://www.youtube.com/watch?v=Ex_ObHVftDg)
Diffie-Hellman Key Exchange

- A classic key exchange protocol.
- Based on modular exponentiation.
- The secret $K_1 = K_2$ shared by Alice and Bob at the end of the protocol would typically be a shared symmetric key.
Diffie-Hellman: Algorithm

1. All parties (set-up):
Diffie-Hellman: Algorithm

1. **All parties** (set-up):
   1. Pick $p$, a prime number.
All parties (set-up):

1. Pick \( p \), a prime number.
2. Pick \( g \), a generator for \( Z_p \).
Diffie-Hellman: Algorithm

1. **All parties** (set-up):
   1. Pick $p$, a prime number.
   2. Pick $g$, a generator for $Z_p$.

2. **Alice**:
Diffie-Hellman: Algorithm

1. **All parties (set-up):**
   1. Pick $p$, a prime number.
   2. Pick $g$, a generator for $\mathbb{Z}_p$.

2. **Alice:**
   1. Pick a random $x \in \mathbb{Z}_p$, $x > 0$. 
Diffie-Hellman: Algorithm

1. **All parties (set-up):**
   1. Pick $p$, a prime number.
   2. Pick $g$, a generator for $\mathbb{Z}_p$.

2. **Alice:**
   1. Pick a random $x \in \mathbb{Z}_p, x > 0$.
   2. Compute
      \[ X = g^x \mod p. \]
Diffie-Hellman: Algorithm

1. **All parties** (set-up):
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   1. Pick a random $x \in \mathbb{Z}_p$, $x > 0$.
   2. Compute
      \[
      X = g^x \mod p.
      \]
   3. Send $X$ to Bob.
Diffie-Hellman: Algorithm

1. **All parties (set-up):**
   1. Pick $p$, a prime number.
   2. Pick $g$, a generator for $\mathbb{Z}_p$.

2. **Alice:**
   1. Pick a random $x \in \mathbb{Z}_p$, $x > 0$.
   2. Compute $X = g^x \mod p$.

3. Send $X$ to Bob.

3. **Bob:**

Diffie-Hellman: Algorithm

1. **All parties** (set-up):
   1. Pick $p$, a prime number.
   2. Pick $g$, a generator for $Z_p$.

2. **Alice**:
   1. Pick a random $x \in Z_p, x > 0$.
   2. Compute
      \[ X = g^x \mod p. \]
   3. Send $X$ to Bob.

3. **Bob**:
   1. Pick a random $y \in Z_p, x > 0$. 
Diffie-Hellman: Algorithm

1. **All parties** (set-up):
   1. Pick \( p \), a prime number.
   2. Pick \( g \), a generator for \( \mathbb{Z}_p \).

2. **Alice**:
   1. Pick a random \( x \in \mathbb{Z}_p, x > 0 \).
   2. Compute
      \[ X = g^x \mod p. \]
   3. Send \( X \) to Bob.

3. **Bob**:
   1. Pick a random \( y \in \mathbb{Z}_p, y > 0 \).
   2. Compute
      \[ Y = g^y \mod p. \]
Diffie-Hellman: Algorithm

1. All parties (set-up):
   1. Pick $p$, a prime number.
   2. Pick $g$, a generator for $\mathbb{Z}_p$.

2. Alice:
   1. Pick a random $x \in \mathbb{Z}_p$, $x > 0$.
   2. Compute
      \[ X = g^x \mod p. \]
   3. Send $X$ to Bob.

3. Bob:
   1. Pick a random $y \in \mathbb{Z}_p$, $x > 0$.
   2. Compute
      \[ Y = g^y \mod p. \]
   3. Send $Y$ to Alice.
Diffie-Hellman: Algorithm

1. **All parties** (set-up):
   1. Pick \( p \), a prime number.
   2. Pick \( g \), a generator for \( \mathbb{Z}_p \).

2. **Alice**:
   1. Pick a random \( x \in \mathbb{Z}_p, x > 0 \).
   2. Compute \( X = g^x \mod p \).
   3. Send \( X \) to Bob.

3. **Bob**:
   1. Pick a random \( y \in \mathbb{Z}_p, x > 0 \).
   2. Compute \( Y = g^y \mod p \).
   3. Send \( Y \) to Alice

4. **Alice** computes the secret: \( K_1 = Y^x \mod p \).
Diffie-Hellman: Algorithm

1. **All parties (set-up):**
   1. Pick $p$, a prime number.
   2. Pick $g$, a generator for $\mathbb{Z}_p$.

2. **Alice:**
   1. Pick a random $x \in \mathbb{Z}_p$, $x > 0$.
   2. Compute
      \[ X = g^x \mod p. \]
   3. Send $X$ to Bob.

3. **Bob:**
   1. Pick a random $y \in \mathbb{Z}_p$, $y > 0$.
   2. Compute
      \[ Y = g^y \mod p. \]
   3. Send $Y$ to Alice

4. **Alice** computes the secret: \[ K_1 = Y^x \mod p. \]

5. **Bob** computes the secret: \[ K_2 = X^y \mod p. \]
Example

Pick \( p = 13 \), a prime number.
Example

1. Pick \( p = 13 \), a prime number.
2. Pick \( g = 2 \), a generator for \( Z_{13} \).
Example

1. Pick $p = 13$, a prime number.
2. Pick $g = 2$, a generator for $\mathbb{Z}_{13}$.
3. Alice:
Example

1. Pick \( p = 13 \), a prime number.
2. Pick \( g = 2 \), a generator for \( \mathbb{Z}_{13} \).
3. Alice:
   1. Pick a random \( x = 3 \).
Example

1. Pick $p = 13$, a prime number.
2. Pick $g = 2$, a generator for $\mathbb{Z}_{13}$.
3. Alice:
   1. Pick a random $x = 3$.
   2. Compute $X = g^x \mod p = 2^3 \mod 13 = 8$. 
Example

1. Pick \( p = 13 \), a prime number.
2. Pick \( g = 2 \), a generator for \( Z_{13} \).
3. Alice:
   1. Pick a random \( x = 3 \).
   2. Compute \( X = g^x \mod p = 2^3 \mod 13 = 8 \).
4. Bob:
Example

1. Pick \( p = 13 \), a prime number.
2. Pick \( g = 2 \), a generator for \( Z_{13} \).
3. Alice:
   1. Pick a random \( x = 3 \).
   2. Compute \( X = g^x \mod p = 2^3 \mod 13 = 8 \).
4. Bob:
   1. Pick a random \( y = 7 \).
Pick $p = 13$, a prime number.

Pick $g = 2$, a generator for $\mathbb{Z}_{13}$.

Alice:
1. Pick a random $x = 3$.
2. Compute $X = g^x \mod p = 2^3 \mod 13 = 8$.

Bob:
1. Pick a random $y = 7$.
2. Compute $Y = g^y \mod p = 2^7 \mod 13 = 11$. 
Example

1. Pick $p = 13$, a prime number.
2. Pick $g = 2$, a generator for $\mathbb{Z}_{13}$.
3. Alice:
   1. Pick a random $x = 3$.
   2. Compute $X = g^x \mod p = 2^3 \mod 13 = 8$.
4. Bob:
   1. Pick a random $y = 7$.
   2. Compute $Y = g^y \mod p = 2^7 \mod 13 = 11$.
5. Alice computes: $K_1 = Y^x \mod p = 11^3 \mod 13 = 5$. 
Example

1. Pick \( p = 13 \), a prime number.
2. Pick \( g = 2 \), a generator for \( \mathbb{Z}_{13} \).
3. **Alice:**
   1. Pick a random \( x = 3 \).
   2. Compute \( X = g^x \mod p = 2^3 \mod 13 = 8 \).
4. **Bob:**
   1. Pick a random \( y = 7 \).
   2. Compute \( Y = g^y \mod p = 2^7 \mod 13 = 11 \).
5. **Alice** computes: \( K_1 = Y^x \mod p = 11^3 \mod 13 = 5 \).
6. **Bob** computes: \( K_2 = X^y \mod p = 8^7 \mod 13 = 5 \).
Example

1. Pick \( p = 13 \), a prime number.
2. Pick \( g = 2 \), a generator for \( \mathbb{Z}_{13} \).
3. **Alice:**
   1. Pick a random \( x = 3 \).
   2. Compute \( X = g^x \mod p = 2^3 \mod 13 = 8 \).
4. **Bob:**
   1. Pick a random \( y = 7 \).
   2. Compute \( Y = g^y \mod p = 2^7 \mod 13 = 11 \).
5. **Alice** computes: \( K_1 = Y^x \mod p = 11^3 \mod 13 = 5 \).
6. **Bob** computes: \( K_2 = X^y \mod p = 8^7 \mod 13 = 5 \).
7. \( \Rightarrow K_1 = K_2 = 5 \).
In-Class Exercise

Let $p = 19$.
Let $g = 10$.
Let Alice’s secret $x = 7$.
Let Bob’s secret $y = 15$.

1. Compute $K_1$.
2. Compute $K_2$. 
Diffie-Hellman Correctness

- Alice has computed

\[ X = g^x \mod p \]
\[ K_1 = Y^x \mod p. \]
Diffie-Hellman Correctness

- Alice has computed

\[ X = g^x \mod p \]
\[ K_1 = Y^x \mod p. \]

- Bob has computed

\[ Y = g^y \mod p \]
\[ K_2 = X^y \mod p. \]
Alice has

\[ K_1 = Y^x \mod p \]
\[ = (g^{y})^x \mod p \]
\[ = (g^x)^y \mod p \]
\[ = X^y \mod p \]
Diffie-Hellman Correctness...

- Alice has

\[ K_1 = Y^x \mod p \]
\[ = (g^y)^x \mod p \]
\[ = (g^x)^y \mod p \]
\[ = X^y \mod p \]

- Bob has

\[ K_2 = X^y \mod p \]
\[ = (g^x)^y \mod p \]
\[ = X^y \mod p \]
Diffie-Hellman Correctness...

- Alice has

\[ K_1 = Y^x \mod p \]
\[ = (g^y)^x \mod p \]
\[ = (g^x)^y \mod p \]
\[ = X^y \mod p \]

- Bob has

\[ K_2 = X^y \mod p \]
\[ = (g^x)^y \mod p \]
\[ = X^y \mod p \]

\[ \Rightarrow K_1 = K_2. \]
Diffie-Hellman Security

- The security of the scheme depends on the hardness of solving the discrete logarithm problem.
Diffie-Hellman Security

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- Generally believed to be hard.
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- **Diffie-Hellman Property**:
Diffie-Hellman Security

- The security of the scheme depends on the hardness of solving the discrete logarithm problem.
- Generally believed to be hard.
- **Diffie-Hellman Property**: Given

\[ p, X = g^x, Y = g^y \]
Diffie-Hellman Security

- The security of the scheme depends on the hardness of solving the discrete logarithm problem.
- Generally believed to be hard.
- **Diffie-Hellman Property:**
  - Given:
    \[ p, X = g^x, Y = g^y \]
  - computing:
    \[ K = g^{xy} \mod p \]
  - is thought to be hard.
Diffie-Hellman: Man-In-The-Middle attack

1. Alice:
Diffie-Hellman: Man-In-The-Middle attack

1 Alice:
   Send $X = g^X \mod p$ to Bob.
Diffie-Hellman: Man-In-The-Middle attack

1. **Alice**: Send $X = g^X \mod p$ to Bob.

2. **Eve**: 

Diffie-Hellman Key Exchange
Diffie-Hellman: Man-In-The-Middle attack

1. **Alice**: Send $X = g^X \mod p$ to Bob.

2. **Eve**: Intercept $X = g^x \mod p$ from Alice.
**Diffie-Hellman: Man-In-The-Middle attack**

1. **Alice:**
   1. Send $X = g^X \mod p$ to Bob.

2. **Eve:**
   1. Intercept $X = g^x \mod p$ from Alice.
   2. Pick a number $t$ in $Z_p$. 

**Diffie-Hellman Key Exchange**
Diffie-Hellman: Man-In-The-Middle attack

1. **Alice:**
   1. Send $X = g^X \mod p$ to Bob.

2. **Eve:**
   1. Intercept $X = g^x \mod p$ from Alice.
   2. Pick a number $t$ in $\mathbb{Z}_p$.
   3. Send $T = g^t \mod p$ to Bob.
Diffie-Hellman: Man-In-The-Middle attack

1. **Alice:**
   1. Send $X = g^X \mod p$ to Bob.

2. **Eve:**
   1. Intercept $X = g^x \mod p$ from Alice.
   2. Pick a number $t$ in $Z_p$.
   3. Send $T = g^t \mod p$ to Bob.

3. **Bob:**
Diffie-Hellman: Man-In-The-Middle attack

1. **Alice:**
   1. Send \( X = g^x \mod p \) to Bob.

2. **Eve:**
   1. Intercept \( X = g^x \mod p \) from Alice.
   2. Pick a number \( t \) in \( Z_p \).
   3. Send \( T = g^t \mod p \) to Bob.

3. **Bob:**
   1. Send \( Y = g^y \mod p \) to Alice.
Diffie-Hellman: Man-In-The-Middle attack

1. **Alice:**
   1. Send $X = g^x \mod p$ to Bob.

2. **Eve:**
   1. Intercept $X = g^x \mod p$ from Alice.
   2. Pick a number $t$ in $\mathbb{Z}_p$.
   3. Send $T = g^t \mod p$ to Bob.

3. **Bob:**
   1. Send $Y = g^y \mod p$ to Alice

4. **Eve:**
Alice:
1. Send $X = g^x \mod p$ to Bob.

Eve:
1. Intercept $X = g^x \mod p$ from Alice.
2. Pick a number $t$ in $Z_p$.
3. Send $T = g^t \mod p$ to Bob.

Bob:
1. Send $Y = g^y \mod p$ to Alice

Eve:
1. Intercept $Y = g^y \mod p$ from Bob.
Diffie-Hellman: Man-In-The-Middle attack

1. **Alice**: Send $X = g^x \mod p$ to Bob.

2. **Eve**: Intercept $X = g^x \mod p$ from Alice.
   - Pick a number $t$ in $\mathbb{Z}_p$.
   - Send $T = g^t \mod p$ to Bob.

3. **Bob**: Send $Y = g^y \mod p$ to Alice.

4. **Eve**: Intercept $Y = g^y \mod p$ from Bob.
   - Pick a number $s$ in $\mathbb{Z}_p$. 
Diffie-Hellman: Man-In-The-Middle attack

1. **Alice:**
   1. Send $X = g^X \mod p$ to Bob.

2. **Eve:**
   1. Intercept $X = g^x \mod p$ from Alice.
   2. Pick a number $t$ in $\mathbb{Z}_p$.
   3. Send $T = g^t \mod p$ to Bob.

3. **Bob:**
   1. Send $Y = g^y \mod p$ to Alice.

4. **Eve:**
   1. Intercept $Y = g^y \mod p$ from Bob.
   2. Pick a number $s$ in $\mathbb{Z}_p$.
   3. Send $S = g^s \mod p$ to Alice.
Diffie-Hellman: Man-In-The-Middle attack.

Alice and Eve:
Diffie-Hellman: Man-In-The-Middle attack.

Alice and Eve:

1. Compute $K_1 = g^{xs} \mod p$
Diffie-Hellman: Man-In-The-Middle attack... 

5 Alice and Eve:
   1 Compute $K_1 = g^{xS} \mod p$

6 Bob and Eve:
Diffie-Hellman: Man-In-The-Middle attack. . .

5 Alice and Eve:
   1 Compute $K_1 = g^{xs} \mod p$

6 Bob and Eve:
   1 Compute $K_2 = g^{yT} \mod p$
Alice: Send $C = E_{K_1}(M)$ to Bob
Diffie-Hellman: Man-In-The-Middle attack. . .

7 Alice: Send $C = E_{K_1}(M)$ to Bob

8 Eve:
Diffie-Hellman: Man-In-The-Middle attack…

7. Alice: Send $C = E_{K_1}(M)$ to Bob

8. Eve:
   - Intercept $C$. 
Diffie-Hellman: Man-In-The-Middle attack...

7 Alice: Send \( C = E_{K_1}(M) \) to Bob

8 Eve:
   1 Intercept \( C \).
   2 Decrypt: \( M = D_{K_1}(C) \)
Diffie-Hellman: Man-In-The-Middle attack...

7 Alice: Send $C = E_{K_1}(M)$ to Bob

8 Eve:

1 Intercept $C$.
2 Decrypt: $M = D_{K_1}(C)$
3 Re-encrypt: $C' = E_{K_2}(M)$
Diffie-Hellman: Man-In-The-Middle attack...

7 Alice: Send $C = E_{K_1}(M)$ to Bob
8 Eve:
   1. Intercept $C$.
   2. Decrypt: $M = D_{K_1}(C)$
   3. Re-encrypt: $C' = E_{K_2}(M)$
   4. Send $C'$ to Bob
Diffie-Hellman: Man-In-The-Middle attack.

7 Alice: Send $C = E_{K_1}(M)$ to Bob
8 Eve:
   1 Intercept $C$.
   2 Decrypt: $M = D_{K_1}(C)$
   3 Re-encrypt: $C' = E_{K_2}(M)$
   4 Send $C'$ to Bob
9 Bob: Send $C = E_{K_2}(M)$ to Alice
Diffie-Hellman: Man-In-The-Middle attack...

1. **Alice:** Send $C = E_{K_1}(M)$ to Bob

2. **Eve:**
   1. Intercept $C$.
   2. Decrypt: $M = D_{K_1}(C)$
   3. Re-encrypt: $C' = E_{K_2}(M)$
   4. Send $C'$ to Bob

3. **Bob:** Send $C = E_{K_2}(M)$ to Alice

4. **Eve:**
### Diffie-Hellman: Man-In-The-Middle attack.

1. **Alice**: Send $C = E_{K_1}(M)$ to Bob

2. **Eve**:
   1. Intercept $C$.
   2. Decrypt: $M = D_{K_1}(C)$
   3. Re-encrypt: $C' = E_{K_2}(M)$
   4. Send $C'$ to Bob

3. **Bob**: Send $C = E_{K_2}(M)$ to Alice

4. **Eve**:
   1. Intercept $C$. 

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**Diffie-Hellman Key Exchange**
Diffie-Hellman: Man-In-The-Middle attack. . .

7. **Alice**: Send $C = E_{K_1}(M)$ to Bob

8. **Eve**:
   1. Intercept $C$.
   2. Decrypt: $M = D_{K_1}(C)$
   3. Re-encrypt: $C' = E_{K_2}(M)$
   4. Send $C'$ to Bob

9. **Bob**: Send $C = E_{K_2}(M)$ to Alice

10. **Eve**:
    1. Intercept $C$.
    2. Decrypt: $M = D_{K_2}(C)$
Diffie-Hellman: Man-In-The-Middle attack...

7 Alice: Send $C = E_{K_1}(M)$ to Bob

8 Eve:
   1. Intercept $C$.
   2. Decrypt: $M = D_{K_1}(C)$
   3. Re-encrypt: $C' = E_{K_2}(M)$
   4. Send $C'$ to Bob

9 Bob: Send $C = E_{K_2}(M)$ to Alice

10 Eve:
   1. Intercept $C$.
   2. Decrypt: $M = D_{K_2}(C)$
   3. Re-encrypt: $C' = E_{K_1}(M)$
Diffie-Hellman: Man-In-The-Middle attack...

7 Alice: Send $C = E_{K_1}(M)$ to Bob

8 Eve:
   1. Intercept $C$.
   2. Decrypt: $M = D_{K_1}(C)$
   3. Re-encrypt: $C' = E_{K_2}(M)$
   4. Send $C'$ to Bob

9 Bob: Send $C = E_{K_2}(M)$ to Alice

10 Eve:
   1. Intercept $C$.
   2. Decrypt: $M = D_{K_2}(C)$
   3. Re-encrypt: $C' = E_{K_1}(M)$
   4. Send $C'$ to Alice.
Outline

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6 Summary
Readings and References

- Chapter 8.1.1-8.1.5 in *Introduction to Computer Security*, by Goodrich and Tamassia.
Acknowledgments

Additional material and exercises have also been collected from these sources:

1. Igor Crk and Scott Baker, 620—Fall 2003—Basic Cryptography.
2. Bruce Schneier, Applied Cryptography.