Tutte Embedding: How to Draw a Graph

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Math 543 Fall 2008

Outline

- Problem definition & Background
- Barycentric coordinates & Definitions
- Tutte embedding motivation
- Barycentric Map Construction
  - Worked example
  - The linear system
- Drawbacks

Problem Definition

- Graph Drawing:

  Given a graph $G = (V, E)$ we seek an injective map (embedding)

$$f: V(G) \longrightarrow \text{Space}$$

  such that G's connectivity is preserved.

For this discussion:

- Space is $\mathbb{R}^2$.
- Edges are straight line segments.

Background

- Early graph drawing algorithms:
  - P. Eades (1984)
  - T. Kamada & S. Kawai (1988)
  - T. Fruchterman & E. Reingold (1991)

- These algorithms are force-directed methods. (a.k.a. spring embedders)
  - Vertices: steel rings
  - Edges: springs
  - Attractive/repulsive forces exist between vertices.
  - System reaches equilibrium at minimum energy.

Background: Tutte Embedding

- **William Thomas Tutte** (May 14, 1917 – May 2, 2002) was a British, later Canadian, mathematician and codebreaker.

William T. Tutte.

- Tutte devised the first known algorithmic treatment (1963) for producing drawings for 3-connected planar graphs.

- Tutte constructed an embedding using barycentric mappings.

- The result is guaranteed to be a plane drawing of the graph.

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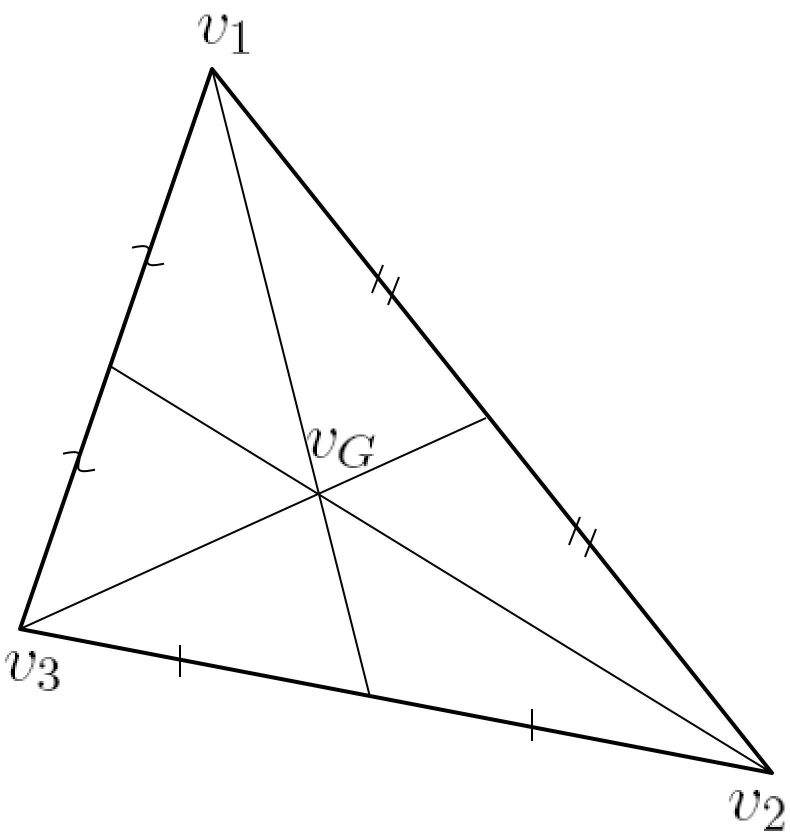
Overview of barycentric coordinates

- Special kind of local coordinates
- Express location of point w.r.t. a given triangle.
- Developed by Möbius in the 19th century.
- Wachspress extended them to arbitrary convex polygons (1975).
- Introduced to computer graphics by Alfeld et al. (1996)

Why *barycentric*?



- v_G is the point where the medians are concurrent.

- v_G is called the barycenter or centroid and in physics it represents the center of mass.

- If $v_G, v_1, v_2, v_3 \in \mathbb{R}^2$ then v_G can be easily calculated as:

$$v_G = \frac{1}{3} \cdot (v_1 + v_2 + v_3)$$

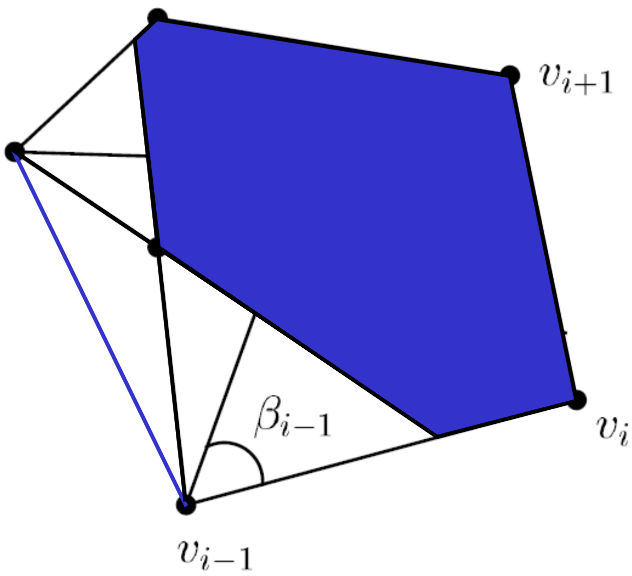
- We want to extend this so that we can express every point $\mathbf{v}$ in terms of the vertices of a polygon $v_1, v_2, \ldots, v_k$.

Convex Combinations

- If P is a polygon with vertices $v_1, v_2, \ldots, v_k \in \mathbb{R}^2$ then we wish to find coordinates $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ such that for $v_0 \in ker(P)$

$$\sum_{i=1}^{k} \lambda_i v_i = v_0$$

- Note that if $\forall i \; \lambda_i > 0$ then v_0 lies inside the convex hull.



Useful definitions

- We say that a representation of G is barycentric relative to a subset J of $V(G)$ if for each v not in J the coordinates $f(v)$ constitute the barycenter of the images of the neighbors of v.

  where $f : V(G) \rightarrow \mathbb{R}^2$

- k-connected graph: If G is connected and not a complete graph, its vertex connectivity $\kappa(G)$ is the size of the smallest separating set in G. We say that G is k-connected if $\kappa(G) \geq$ k.

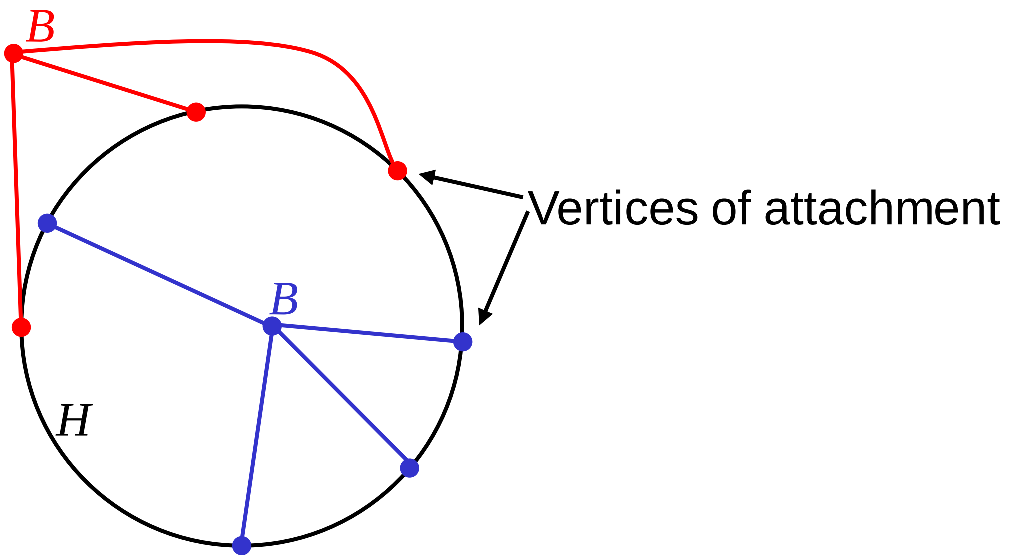
  e.g. The minimum cardinality of the separating set of a 3-connected graph is 3.

Useful definitions(2)

- Given $H \leq_S G$, define relation ~ on $E(G)$-$E(H)$:
  $e \sim e_0$ if ∃walk w starting with e, ending with e_0, s.t. no internal vertex of w is in H.

- Bridge: a subgraph B of G-$E(H)$ if it is induced by ~.

- A peripheral polygon: A polygonal face P of G is called peripheral if P has at most 1 bridge in G.



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Tutte embedding motivation

- The idea is that if we can identify a peripheral P then its bridge B (if is exists) always avoids “all other bridges”… (True—there aren’t any others!)

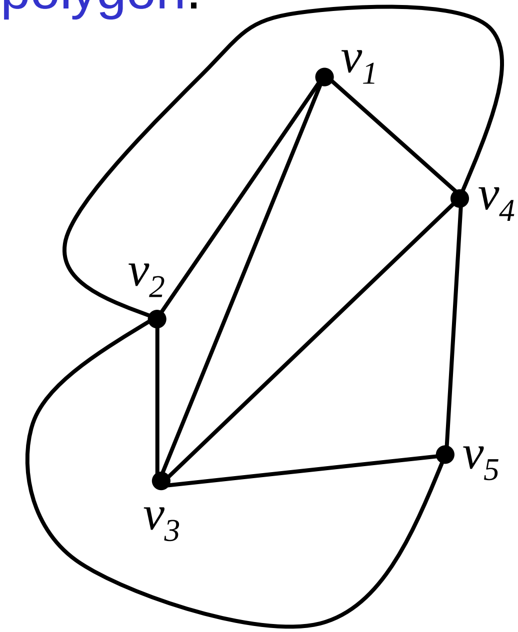
- This means the bridge is transferable to the interior region and hence P can act as the fixed external boundary of the drawing.

- All that remains then is the placement of the vertices in the interior.

Tutte embedding motivation(2)

- **Theorem**: If M is a planar mesh of a nodally 3-connected graph G then each member of M is peripheral.

- In other words, Tutte proved that any face of a 3-connected planar graph is a peripheral polygon.



- This implies that when creating the embedding we can pick any face and make it the outer face (convex hull) of the drawing.

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Barycentric mapping construction

- Steps:

1. Let J be a peripheral polygon of a 3-connected graph G with no Kuratowski subgraphs ($K_{3,3}$ and K_5).
2. We denote the set of nodes of G in J by $V(J)$, and $|V(J)| = n$.

   Suppose there are at least 3 nodes of G in the vertex set of J.
3. Let Q be a geometrical n-sided convex polygon in Euclidean plane.
4. Let f be a 1-1 mapping of $V(J)$ onto the set of vertices of Q s.t. the cyclic order of nodes in J agrees, under f, with the cyclic order of vertices of Q.
5. We write $m= |V(G)|$ and enumerate the vertices of G as v_1, v_2, v_3, …, v_m so the first n are the nodes of G in J.
6. We extend f to the other vertices of G by the following rule.

   If $n < i \leq m$ let $N(i)$ be the set of all vertices of G adjacent to v_i

Barycentric mapping construction(2)

6. For each v_i in $N(i)$ let a unit mass m_j to be placed at the point $f(v_i)$. Then $f(v_i)$ is required to be the centroid of the masses m_j.

7. To investigate this requirement set up a system of Cartesian coordinates, denoting the coordinates of $f(v_i)$, $1 \le i \le m$, by (v_{ix}, v_{iy}).

8. Define a matrix $K(G) = \{C_{ij}\}$, $1 \le (i,j) \le m$, as follows.
   - If $i \neq j$ then C_{ij} = -(number of edges joining v_i and v_j)
   - If $i = j$ then $C_{ij} = deg(v_i)$

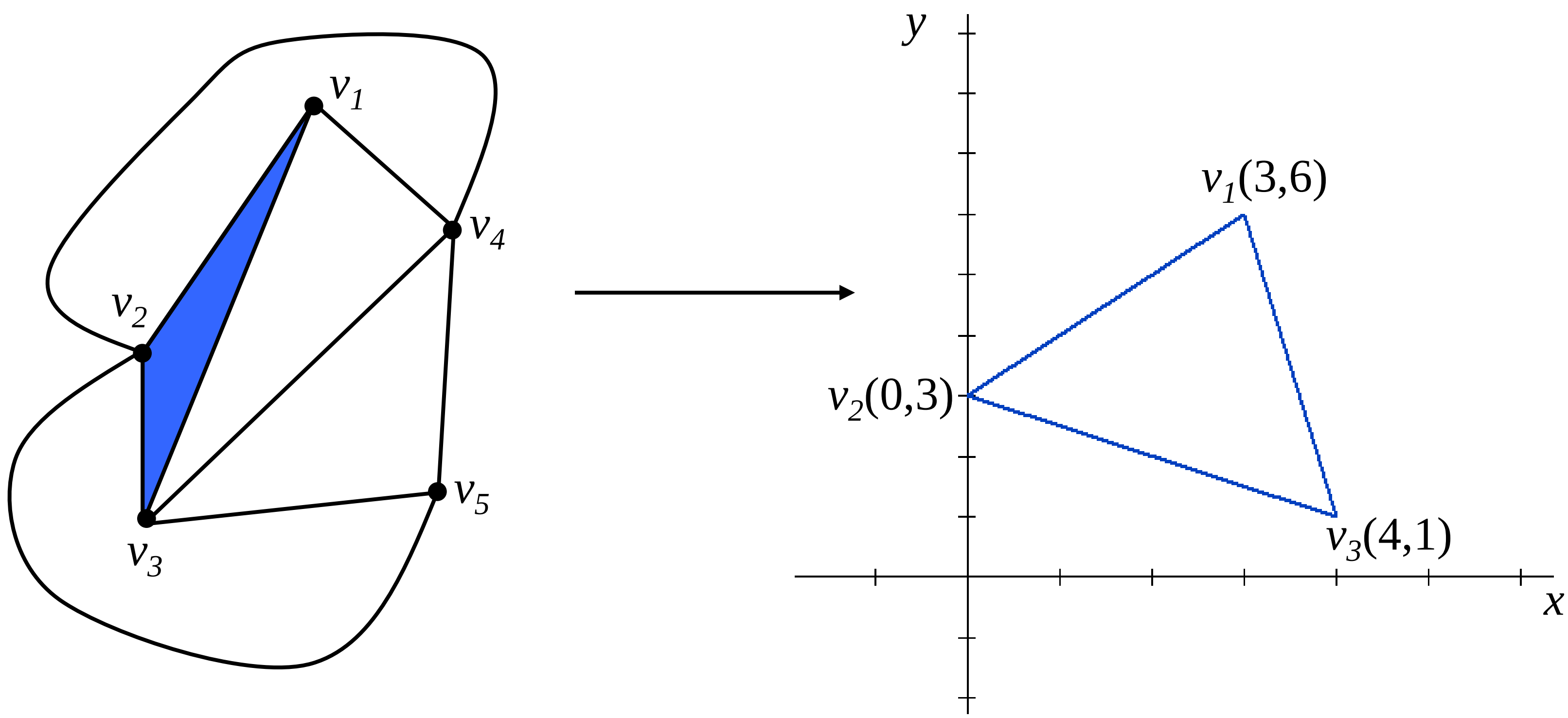
9. Then the barycentric requirement specifies coordinates v_{ix}, v_{iy} for $n < j \le m$ as the solutions to the two linear systems

$$\sum_{j=1}^{m} C_{ij} v_{ix} = 0 \qquad \sum_{j=1}^{m} C_{ij} v_{iy} = 0$$

where $n < i \le m$. For $1 \le j \le n$ the coordinates are already known.

Example

G:

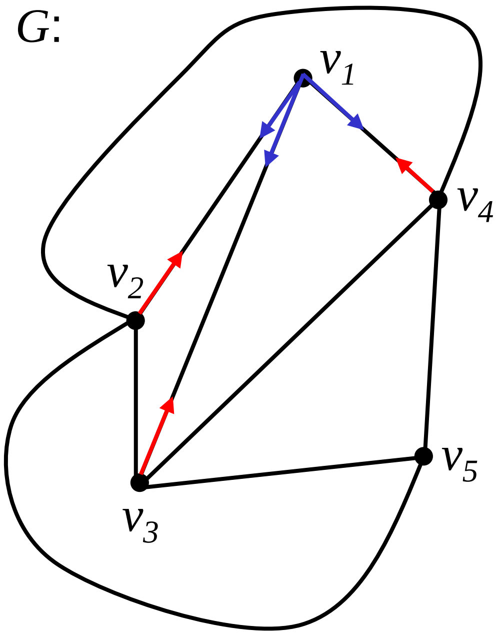


G is 3-connected with unique cut set $\{v_2, v_3, v_4\}$

Consider the peripheral cycle J , $V(J) = \{v_1, v_2, v_3\}$

Example(2)

- $V(J) = \{v_1, v_2, v_3\}$
- $N(4) = \{v_1, v_2, v_3, v_5\}$
- $N(5) = \{v_2, v_3, v_4\}$
- $K(G) = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{pmatrix}$



- Form the 2 linear systems for $i = 4, 5$.

Example(3)

- The linear systems

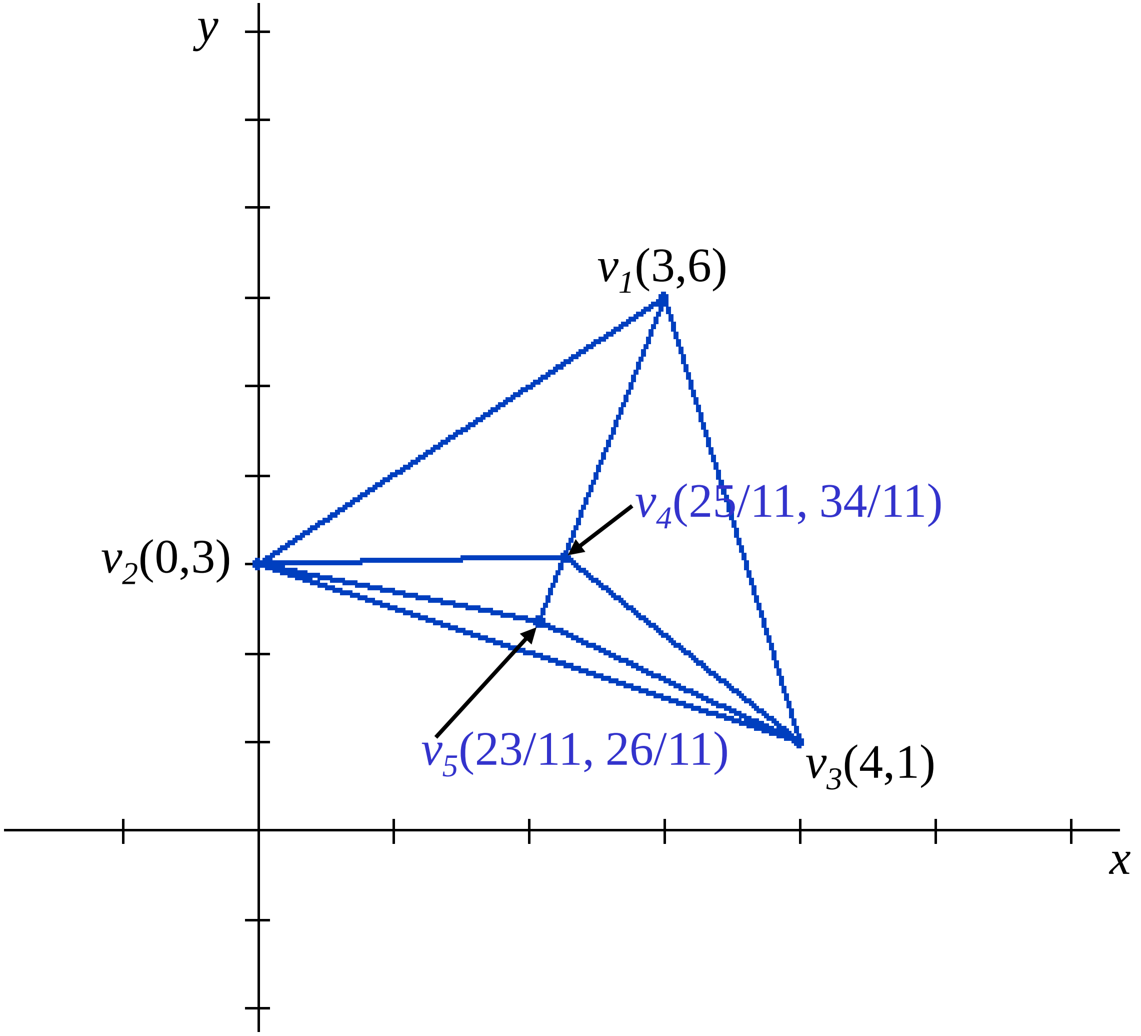
$$C_{41} v_{1x} + C_{42} v_{2x} + C_{43} v_{3x} + C_{44} v_{4x} + C_{45} v_{5x} = 0 \rightarrow 4v_{4x} - 7 = v_{5x}$$
$$C_{51} v_{1x} + C_{52} v_{2x} + C_{53} v_{3x} + C_{54} v_{4x} + C_{55} v_{5x} = 0 \rightarrow -v_{4x} + 3 v_{5x} = 4$$

$$C_{41} v_{1y} + C_{42} v_{2y} + C_{43} v_{3y} + C_{44} v_{4y} + C_{45} v_{5y} = 0 \rightarrow 4v_{4y} - v_{5y} = 10$$
$$C_{51} v_{1y} + C_{52} v_{2y} + C_{53} v_{3y} + C_{54} v_{4y} + C_{55} v_{5y} = 0 \rightarrow -v_{4y} + 3 v_{5y} = 4$$

- Solutions

$v_4(25/11, 34/11)$ $v_5(23/11, 26/11)$

Example: Tutte embedding



The linear system

- Is the linear system always consistent?
- Yes, it is!
- **Proof**:
  - Recall matrix $K(G)$.
    It was defined as $K(G) = \{C_{ij}\}$, $1 \leq (i,j) \leq m$.
    - If $i \neq j$ then C_{ij} = -(number of edges joining v_i and v_j)
    - If $i = j$ then $C_{ij} = deg(v_i)$
  - Observe that this means we can write $K(G)$ as

$$K(G) = -A+D$$

    - where A is the adjacency matrix of G and
    - D is diagonal matrix of vertex degrees.
  - But that's the Laplacian of G! i.e., $K = -L$.

The linear system(2)

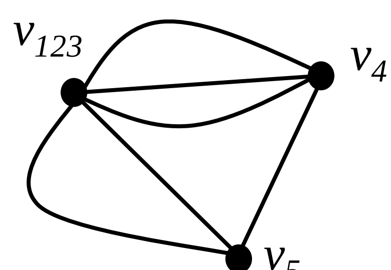
– Let K_1 be the matrix obtained from $K(G)$ by striking out the first n rows and columns.

e.g.

$$K(G) = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{pmatrix} \qquad K_1 = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$$

– Let G_0 be the graph obtained from G by contracting all the edges of J while maintaining the degrees.

G_0:

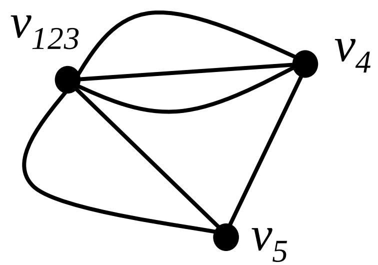


The linear system(3)

For a suitable enumeration of $V(G_0)$, K_1 is obtained from $K(G_0)$ by striking out the first row and column.

$$-L(G_0) = K(G_0) = \begin{pmatrix} 5 & -3 & -2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{pmatrix}$$

G_0:



That is, $K_1 = -\hat{L}_{11}$.

But then the $det(K_1) = det(-\hat{L}_{11}) = t(G)$ is the number of spanning trees of G_0.

$$det(-\hat{L}_{11}) = \begin{vmatrix} 4 & -1 \\ -1 & 3 \end{vmatrix} = 11$$

The linear system(4)

- The number $t(G)$ is non-zero since G_0 is connected.
  - Edge contraction preserves connectedness.

- This implies that $det(K_1) \neq 0$ and the hence the linear systems always have a unique solution.

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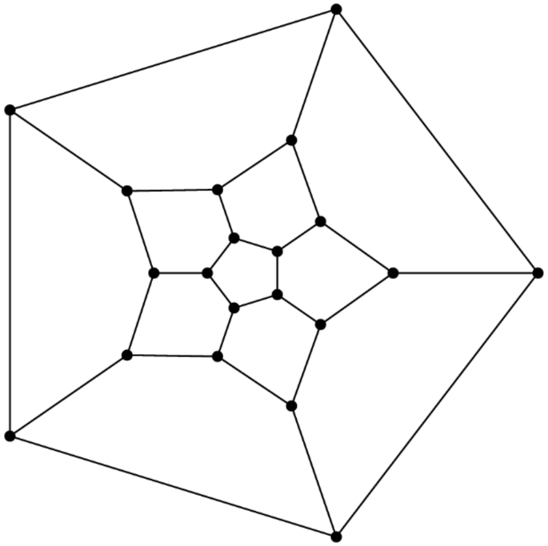
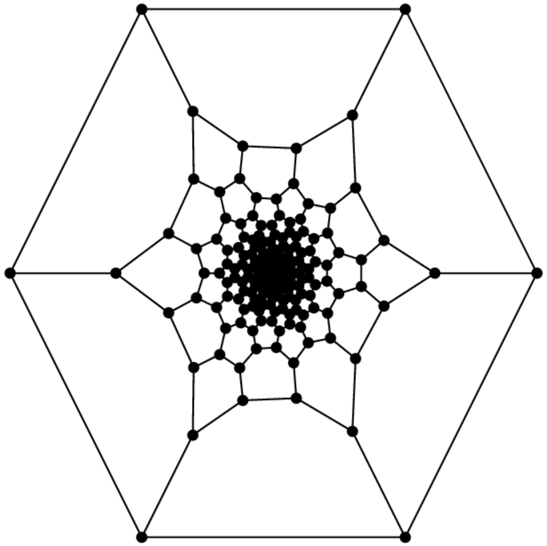
Drawbacks of Tutte Embedding

Only applies to 3-connected planar graphs.

Works only for small graphs (|V| < 100).

The resulting drawing is not always "aesthetically pleasing."





Tutte representation: Dodecahedron *Le*(*C*60)

References

- Alen Orbanić, Tomaž Pisanski, Marko Boben, and Ante Graovac. "Drawing methods for 3-connected planar graphs." *Math/Chem/Comp 2002*, Croatia 2002.
- William T. Tutte. "How to draw a graph." *Proc. London Math. Society*, 13(52):743–768, 1963.

Thank you!

Questions?