CSC 580 Principles of Machine Learning

16 Reinforcement learning (RL)

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*slides credit: built upon CSC 580 Fall 2021 lecture slides by Kwang-Sung Jun & Chicheng Zhang

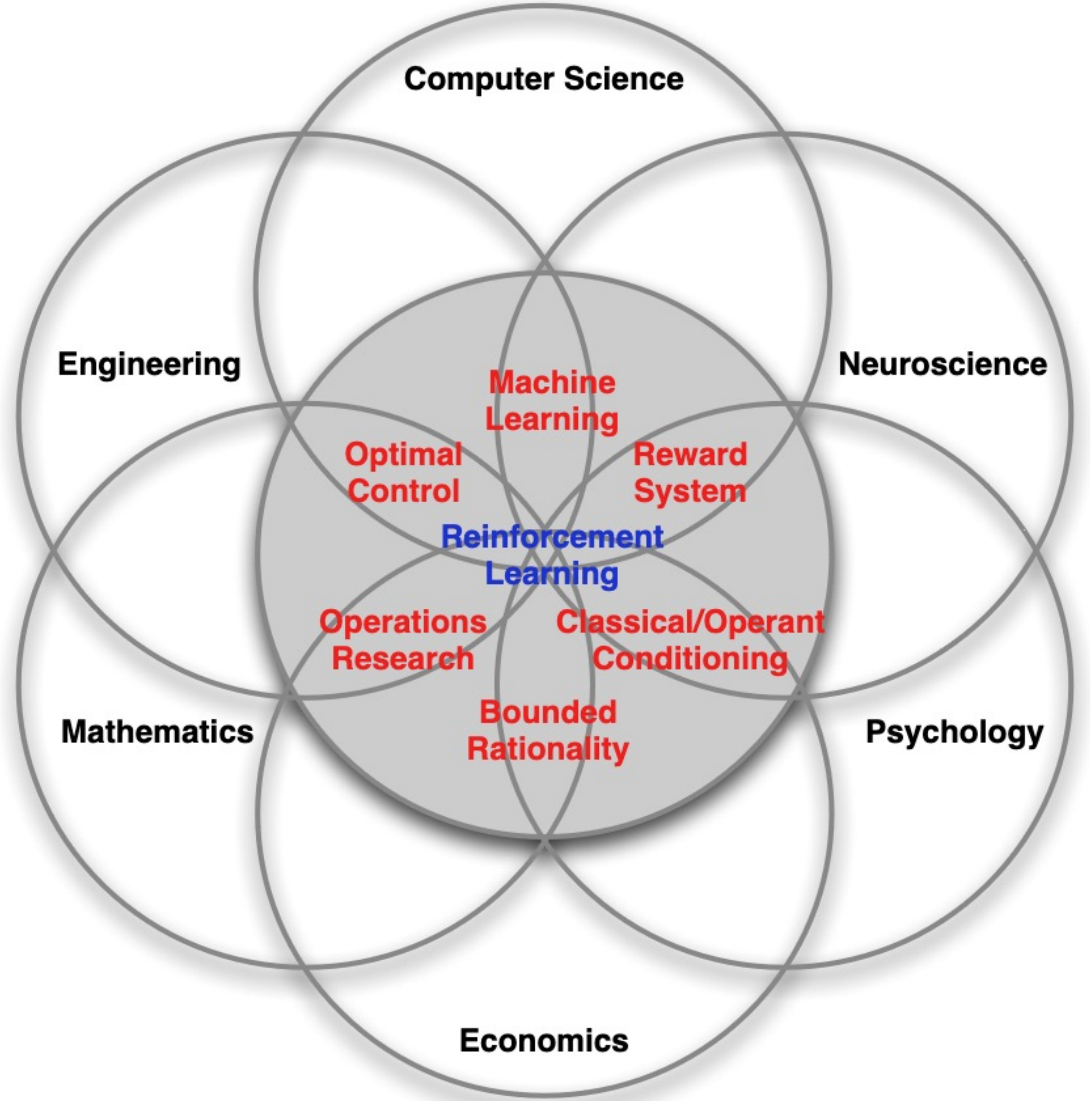
Reinforcement learning references

- ‘’Reinforcement learning’’ by Sutton & Barto (available online)
- RL course by David Silver: https://www.youtube.com/watch?v=2pWv7GOvuf0&list=PLzuuYNsE1EZAXYR4FJ75jcJseBmo4KQ9-

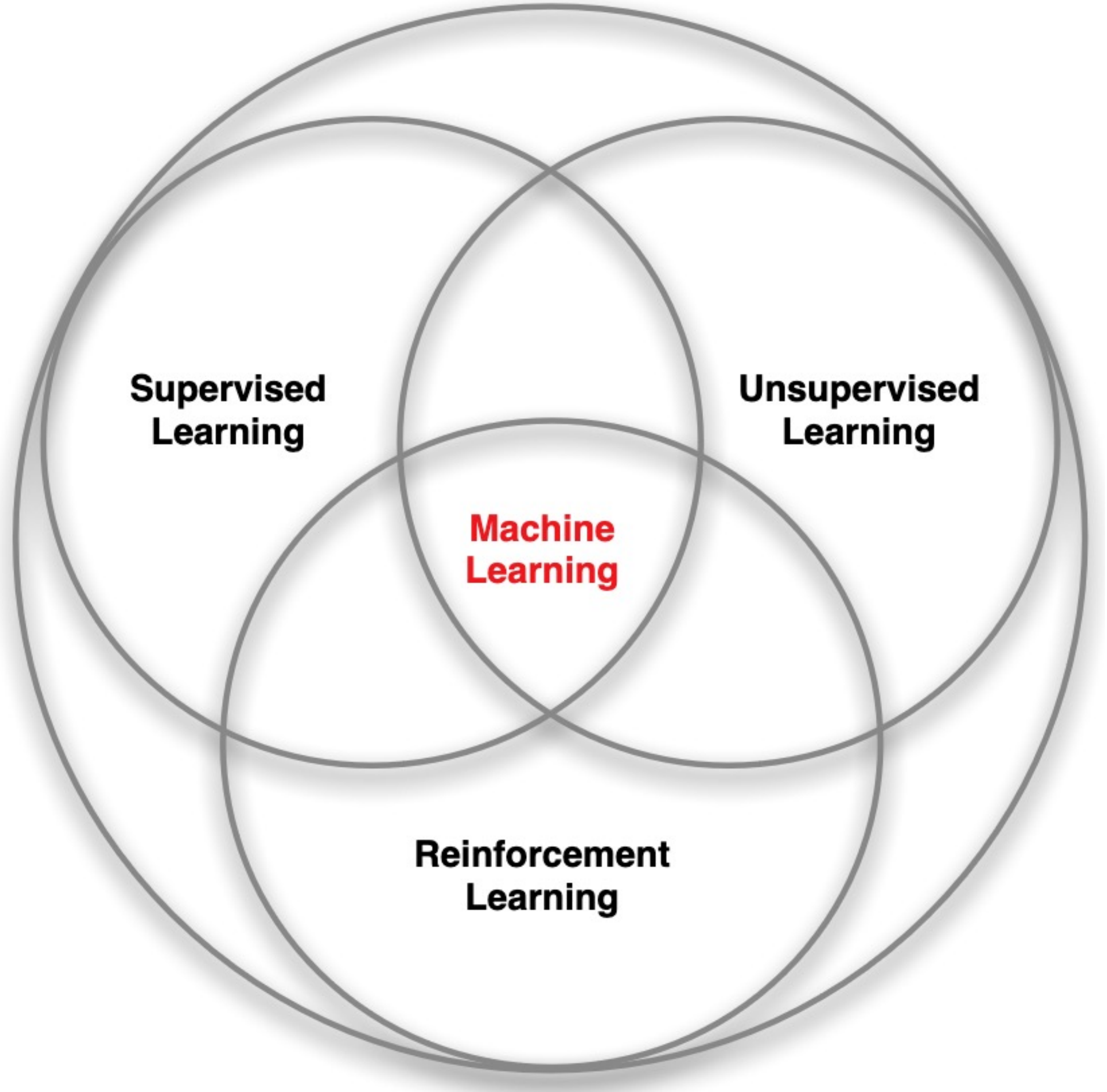
Outline

- Background / Markov Decision Processes (MDPs)
- Planning in MDPs
- Reinforcement Learning in MDPs

Background / Markov Decision Processes



Source: David Silver



Source: David Silver

Reinforcement Learning (RL)

- Task of an agent embedded in an environment
- repeat forever:
  - 1) sense world (=state)
  - 2) reason
  - 3) take an action (this changes the state)
  - 4) get feedback (usually a real-valued **<u>reward</u>**),
  - 5) learn from the feedback


Characteristics of RL

How does RL differ from other ML frameworks?

- There is no supervisor, only a reward signal
- Feedback is not instantaneous (decisions lead to delayed reward)
- Data is not i.i.d. (it is sequential, time matters)
- The agent’s actions affect subsequent data it receives

Source: David Silver

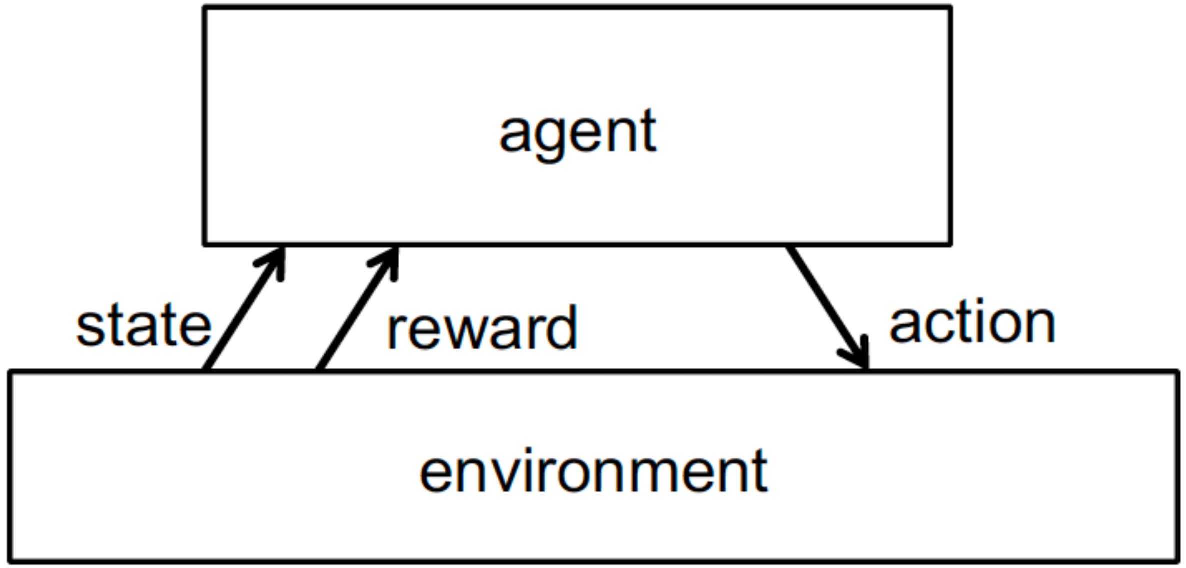
Examples of RL

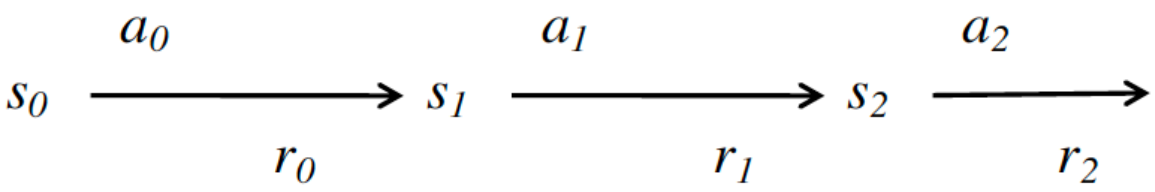
- Fly stunt maneuvers in a helicopter (reward: not crashing)
- Manage an investment portfolio (reward: $)
- Play many different video games (reward: score)
- Make a humanoid robot walk (reward: distance traveled)
- Defeat world champion in Backgammon (reward: win/lose)
- Defeat world champion in Go! (reward: win/lose)

Examples

- https://www.youtube.com/watch?v=TmPfTpjtdgg
- https://www.youtube.com/watch?v=0JL04JJjocc
- https://www.youtube.com/watch?v=gn4nRCC9TwQ

Markov Decision Process (MDP)





- Environment model $\mathcal{M}$
- Set of states S
- Set of actions A
- at each time t, agent observes state $s_t \in S$, then chooses action $a_t \in A$
- then receives a reward r_t and moves to state s_{t+1}; repeat.

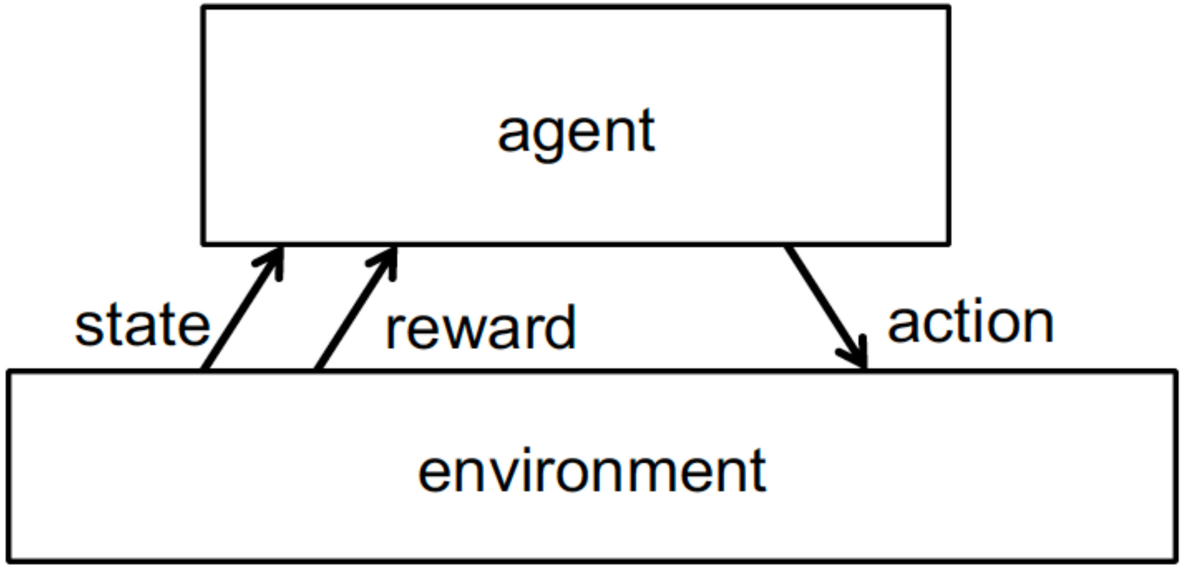
Markov Decision Process (MDP)

Markov assumption:

$$P(r_t|s_t, a_t, s_{t-1}, a_{t-1}, \dots) = P(r_t|s_t, a_t)$$
$$P(s_{t+1}|s_t, a_t, s_{t-1}, a_{t-1}, \dots) = P(s_{t+1}|s_t, a_t)$$

These are unknown to the learner!

i.e. the future is independent of the past, given the present

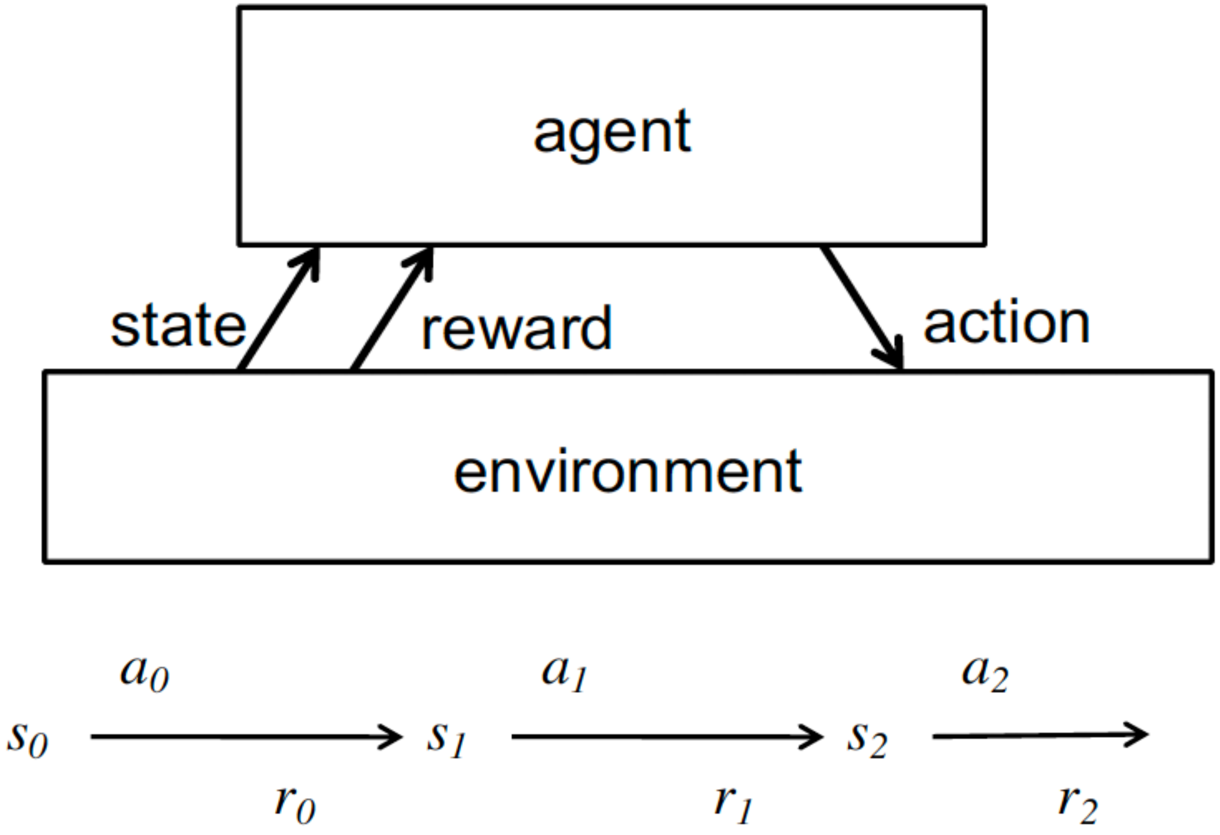


$$s_0 \xrightarrow[r_0]{a_0} s_1 \xrightarrow[r_1]{a_1} s_2 \xrightarrow[r_2]{a_2}$$

Markov Decision Process (MDP)

- A **policy** is the agent's behavior
- It is a map from state to action, e.g.
- Deterministic policy: $a = \pi(s)$
- Stochastic policy:

$$\pi(\, a \mid s \,) = P(A_t = a \mid S_t = s)$$

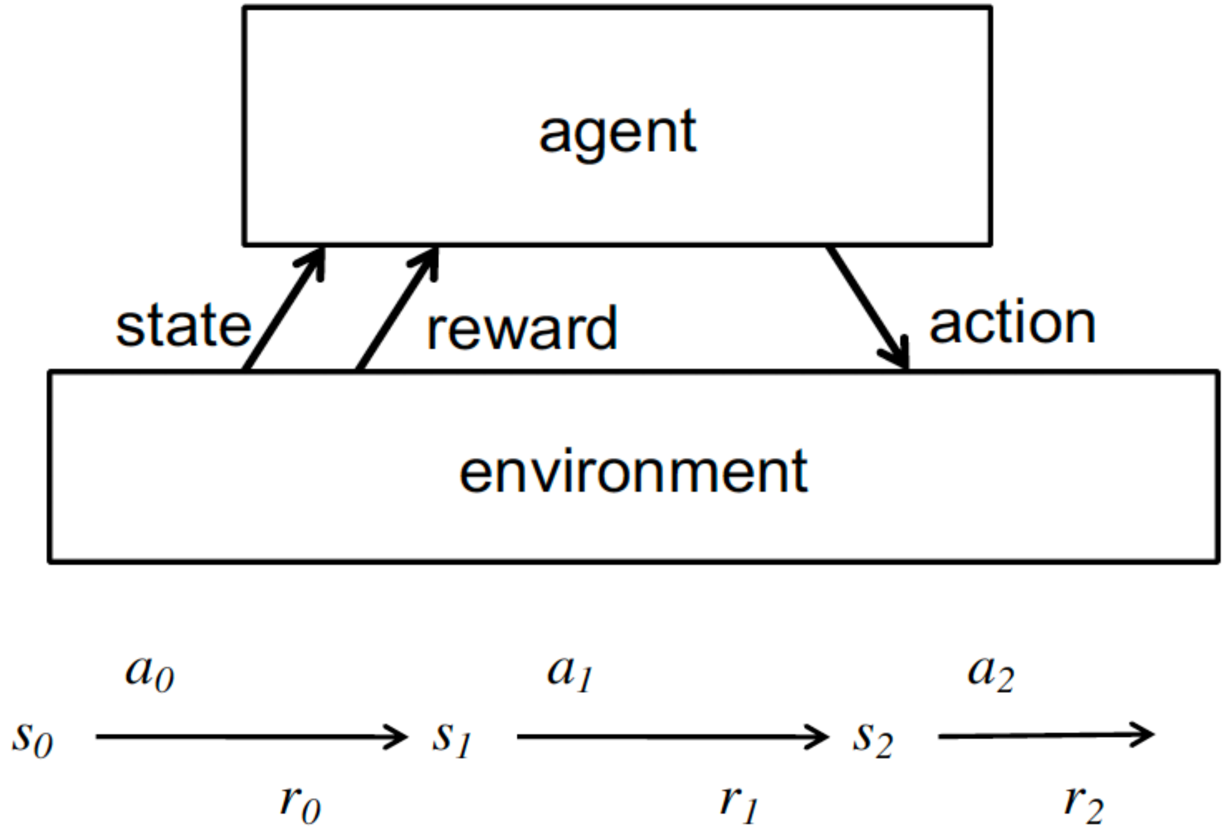


Markov Decision Process (MDP)

Goal:
Learn a <u>policy</u> $\pi: S \to A$ for choosing actions that maximizes **<u>expected cumulative (discounted) reward</u>**

$$\mathbb{E}_{\pi}[r_0 + \gamma\, r_1 + \gamma^2 r_2 + \cdots \mid s_0] \text{ where } 0 \leq \gamma < 1$$

for every possible starting state s_0



The intention behind the RL formulation

- Note that the formulation is **reward-driven.**
- Example: Robot learning: move a dish from one place to another
  - We can assign reward +10 when it accomplishes the task
  - We can also assign reward +1 when it picks up the dish successfully
- *Evaluative* feedback (cf. Instructive feedback – supervised learning)

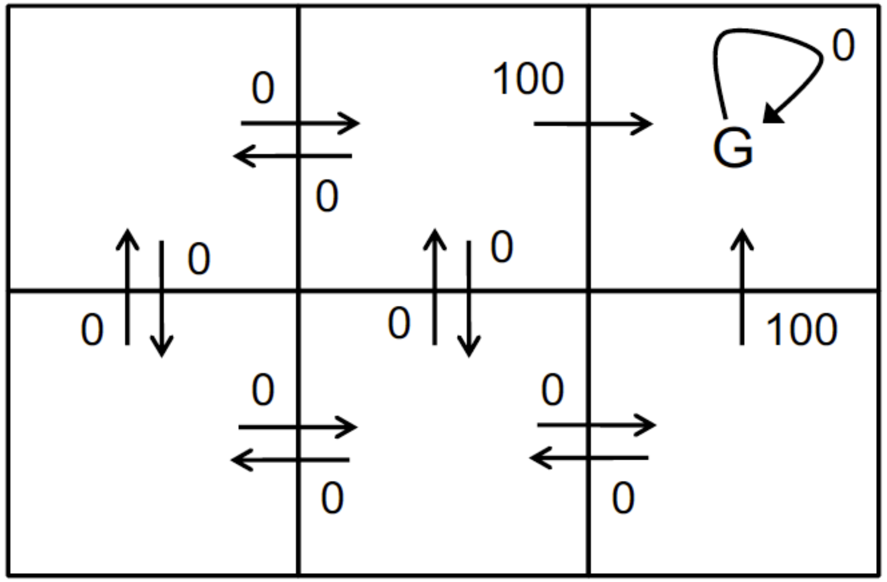
Main Hypothesis:
All goals can be described by the maximization of expected cumulative reward.

(from David Silver's lecture)

Goal	Reward
Walk	Forward displacement
Escape maze	-1 if not out yet; 0 if out
Robots for recycling soda cans	+1 if a new can collected; -10 if run into things; 0 otherwise.
Win chess	0 if not finished; +1 if win; -1 if lose

The grid world: Learning to Navigate

- The grid world



- State s: the location of the agent
- Each arrow represents an **action** a and the associated number represents **reward** $r(s, a)$ (assume that it is deterministic for now).

The structure of returns

- Define return at time step t:

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots$$

- The goal of RL: find a policy π that maximizes its return at the start:

$$\mathbb{E}_\pi[r_0 + \gamma\, r_1 + \gamma^2 r_2 + \cdots] = \mathbb{E}_\pi[G_0]$$

- G_t satisfies the following recurrence:

$$G_t = r_t + \gamma(r_{t+1} + \gamma r_{t+2} + \cdots) = r_t + \gamma G_{t+1}$$

Current return Immediate reward Future return

Value Function

- Prediction of future reward
- Used to evaluate goodness / badness of states
- And therefore, to select actions, e.g.

$$V^{\pi}(s) = \mathbb{E}[\, r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots \mid s_t = s, \pi \,]$$

- We explicitly notate that the value depends on the policy

Value function for a policy

- Given a policy $\pi: S \rightarrow A$, define its *value function* $V^\pi(s) = \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t r_t \mid s_0 = s, \pi]$
- Important property (Bellman consistency equation):

$$
\begin{aligned}
V^\pi(s) &= \mathbb{E}[G_0 \mid s_0 = s, \pi] \\
&= \mathbb{E}[r_0 \mid s_0 = s, \pi] + \gamma \mathbb{E}[G_1 \mid s_0 = s, \pi] \\
&= R(s, \pi(s)) + \gamma \mathbb{E}_{s' | s, \pi(s)}[V^\pi(s')]
\end{aligned}
$$

where $R(s, a) = \mathbb{E}[r_t \mid s_t = s, a_t = a]$

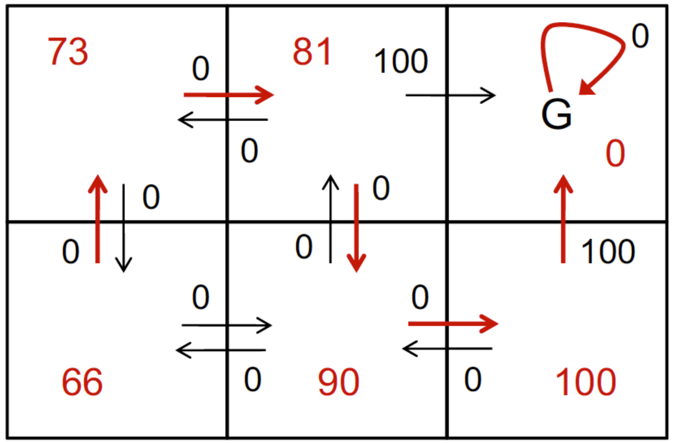
- Fact: there is a policy π^* such that $\pi^* = \arg\max_\pi V^\pi(s)$ for all s
    - π^* is called the *optimal policy*
- $V^*(s)$:= the value function achieved by the optimal policy – optimal value function

* Note: We assume deterministic policies for simplicity; nondeterministic policy would assign probabilities to actions given state; i.e., p(a|s) =: $\pi(a|s)$ => $V^\pi(s) = \sum_{a \in A} \pi(a|s)\big(R(s, a) + \gamma \mathbb{E}_{s'|s,a}[V^\pi(s')]\big)$

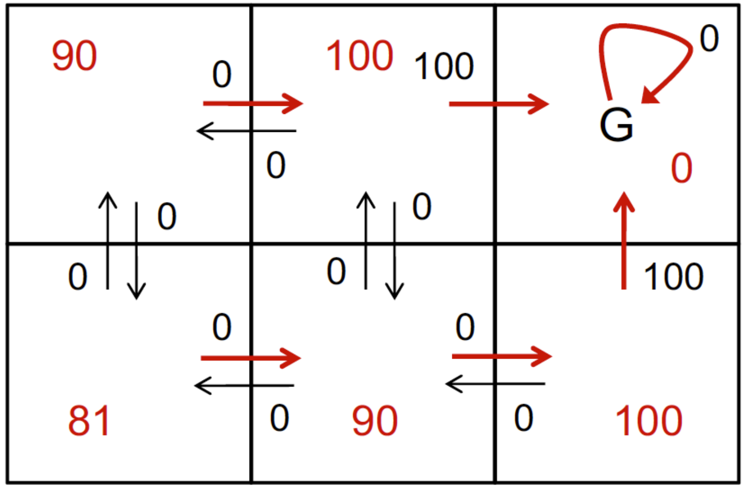
Value function for a policy π

- Suppose π is shown by red arrows, $\gamma = 0.9$

  $V^{\pi}(s)$ values are shown in red



optimal policy π^*



- The Bellman consistency equation:

$$V^{\pi}(s) = R(s, \pi(s)) + \gamma \cdot \sum_{s'} P(s'|s, \pi(s))\, V^{\pi}(s')$$

* stochastic policy: $V^{\pi}(s) = \sum_a \pi(a|s) \left(R(s,a) + \gamma \cdot \sum_{s'} P(s'|s,a)\, V^{\pi}(s')\right)$

Policy evaluation

- How to compute V^π given MDP $\mathcal{M}$ and policy π?
- Recall Bellman consistency equation:

$$V^\pi(s) = \sum_a \pi(a|s)\left(R(s,a) + \gamma \cdot \sum_{s'} P(s'|s,a)\, V^\pi(s')\right)$$
$$= \underbrace{\sum_a \pi(a|s) R(s,a)}_{R^\pi(s)} + \gamma \cdot \sum_{s'} \underbrace{\left(\sum_a \pi(a|s)\, P(s'|s,a)\right)}_{M^\pi(s,s')} V^\pi(s')$$

- In matrix form (denote by $V^\pi = \left(V^\pi(s)\right)_{s\in S} \in \mathbb{R}^{|S|}$, etc):

$$V^\pi = R^\pi + \gamma M^\pi V^\pi$$

(recall the vector/matrix notation here)

- A linear system! How to solve it?
  - Gaussian elimination
- Is this efficient?
  - Time complexity: $O(|S|^3)$

Policy evaluation (cont'd)

$$V^\pi(s) = \sum_a \pi(a|s) \left(R(s,a) + \gamma \cdot \sum_{s'} P(s'|s,a)\, V^\pi(s') \right)$$

<u>Fixed point iteration for policy evaluation</u>

Initialize: V^π arbitrarily (e.g., all zero).

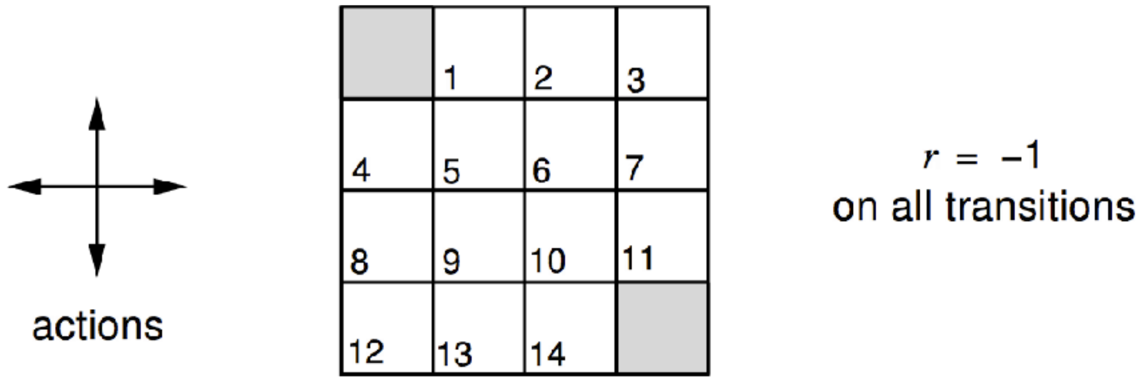
- While V^π does not change much from the previous iteration
  - $W^\pi \leftarrow V^\pi$
  - For each $s \in S$
    - $V^\pi(s) \leftarrow \sum_a \pi(a|s) \left(R(s,a) + \sum_{s'} P(s'|s,a) \cdot \gamma\, W^\pi(s') \right)$

- This is called synchronous update

- Asynchronous update: remove $W^\pi \leftarrow V^\pi$ and perform in-place updates for V^π
  - Preferred method.

Fixed point iteration: an illustration

- Episodic MDP (i.e., terminal states involved) with $\gamma = 1$
- Shaded squares are **terminal states**
- 4 actions
- Actions to the wall end up with the same state.
- Rewards are -1 until the terminal state is reached.
- The policy π: take an action uniformly at random.



Side Q: what's the optimal policy under this reward setting?

Example

- Synchronous updates.
- Values are propagated!

v_k for the Random Policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

$$V^{\pi}(s) \leftarrow \sum_{a} \pi(a|s) \left(R(s,a) + \gamma \sum_{s'} P(s'|s,a) \cdot W^{\pi}(s') \right)$$

Planning in MDPs

Planning in MDPs

- Given: full specification of $\mathcal{M}$, (specifically $\boldsymbol{R(s,a)}$ **and** $\boldsymbol{P(s'|s,a)}$ are known)
- Goal: find optimal policy π^* of $\mathcal{M}$

- Recall: $V^*(s)$ is the value function of the optimal policy.
- Claim: To find the optimal policy, it suffices to find $V^*(s)$ for every state s
- Why?

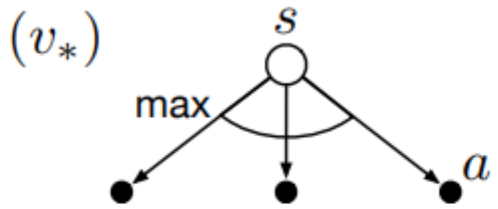
$$\pi^*(s_t) = \arg\max_{a \in A} R(s_t, a) + \gamma \sum_{s \in S} P(s_{t+1} = s | s_t, a) V^*(s)$$

- How to find $V^*(s)$?

Bellman optimality equation

- Fact: $V^*(s) = \max_\pi V^\pi(s)$ satisfies the following equation:

$$V^*(s) = \max_a \left(R(s,a) + \gamma \cdot \sum_{s'} P(s'|s,a)\, V^*(s') \right)$$



- This is known as the Bellman optimality equation
- Intuition:
  - $R(s,a) + \gamma \cdot \sum_{s'} P(s'|s,a)\, V^*(s')$ is the return achieved by: (1) taking action a; and (2) behave optimally afterwards
  - Optimal behavior = optimal action a + optimal behavior afterwards

- Issue: Bellman optimality equation has **no closed form solution**. (unlike computing V^π!)
- However, V^* can still be seen as a fixed point

Algorithm: Value iteration

Key idea: perform fixed point iteration on Bellman optimality equation

$$\boxed{V^*(s) = \max_a \left(R(s,a) + \gamma \cdot \sum_{s'} P(s'|s,a)\, V^*(s') \right)}$$

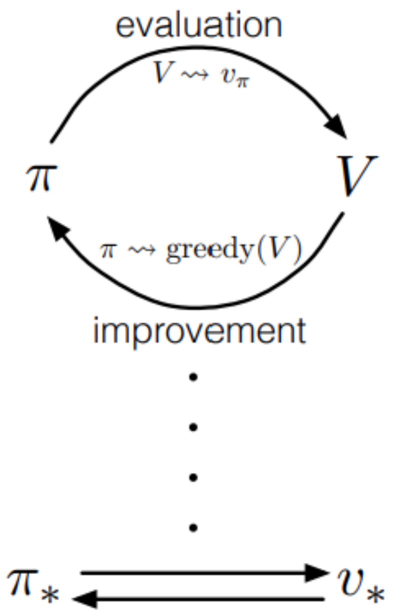
Initialize $V(s)$ arbitrarily

While $\{V(s)\}_{s \in S}$ is not much different from the previous iteration's $\{V(s)\}_{s \in S}$

- **For** each $s \in S$
  - $V(s) \leftarrow \max_a \; R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) \cdot V(s')$
- **End For**

End While

Algorithm: Policy iteration



- The idea:

  estimate optimal value V^* and optimal policy π^* *simultaneously & iteratively*

- Observe:
  - π^* is greedy wrt V^*
  - V^* is the value function of π^*
- Can we obtain a pair (π, V) that exhibit the above properties?

Algorithm:

- Start from an arbitrary policy π (e.g., assign actions randomly)
- Repeat the following:
  - **[Policy evaluation]** $V \leftarrow V^\pi$ (either solve the linear system or iterative method)
  - **[Policy improvement]** Update the policy: $\pi \leftarrow \text{greedy}(V)$

    For every s $\in S$, $\pi(s) \leftarrow \arg\max_a r(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^\pi(s')$

Policy iteration with inexact policy evaluation

Suppose we perform fixed-point iteration for evaluating V^{π}, with $\pi(a \mid s) = 1/4, \forall s, a$

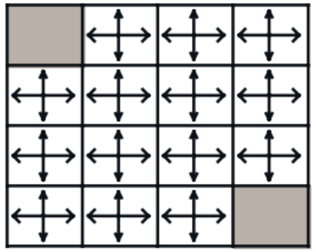
what you get if you apply the policy improvement step

v_k for the Random Policy

Greedy Policy w.r.t. v_k

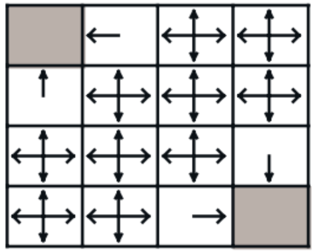
$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



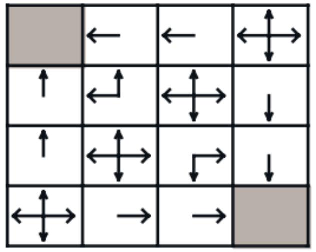
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0



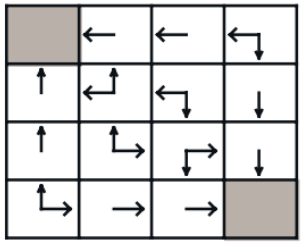
$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0



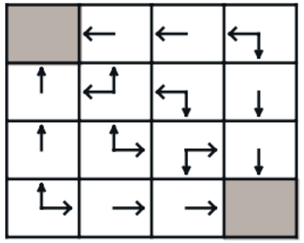
$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0



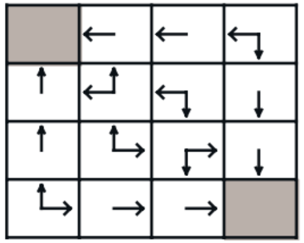
$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0



$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



optimal policy

Algorithm: Modified policy iteration

- From previous slide: inexact value functions are still useful!

- Start from an arbitrary policy π (e.g., assign actions randomly)

- **[(Inexact) Policy evaluation]** $V \leftarrow$ take k fixed-point iterations for computing V^{π} (so $V \approx V^{\pi}$)

  This is **not a valid value function** anymore (no corresponding π that achieves this value in general)

- **[Policy improvement]** Update the policy:
  - For every s $\in S$, $\pi(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V(s')$

Summary

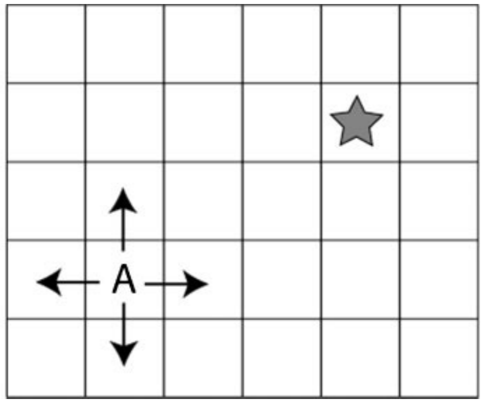
- Policy evaluation: just evaluates the value function for a given π
  - closed form / fixed-point iteration

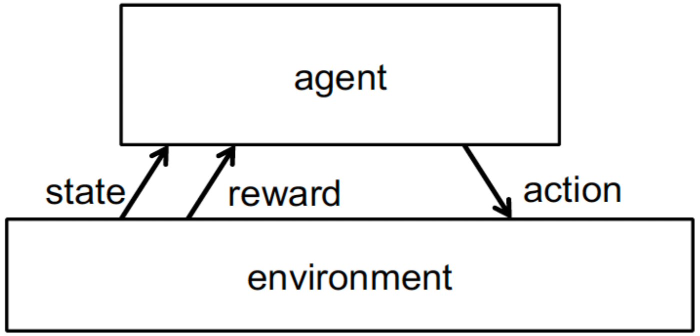
- Planning:
  - Policy iteration: policy evaluation + policy improvement
  - Modified policy iteration: only k steps of policy evaluation
  - Value iteration: k=1

- Recall: so far, we are in the **planning** setting, where we are already given a **model** of the world: i.e. know $P(s'|s,a)$ and $P(r \mid s,a)$
- What if we don't? This is called the **"learning in MDPs"** problem

Learning in MDPs

Learning in MDPs: basic setup



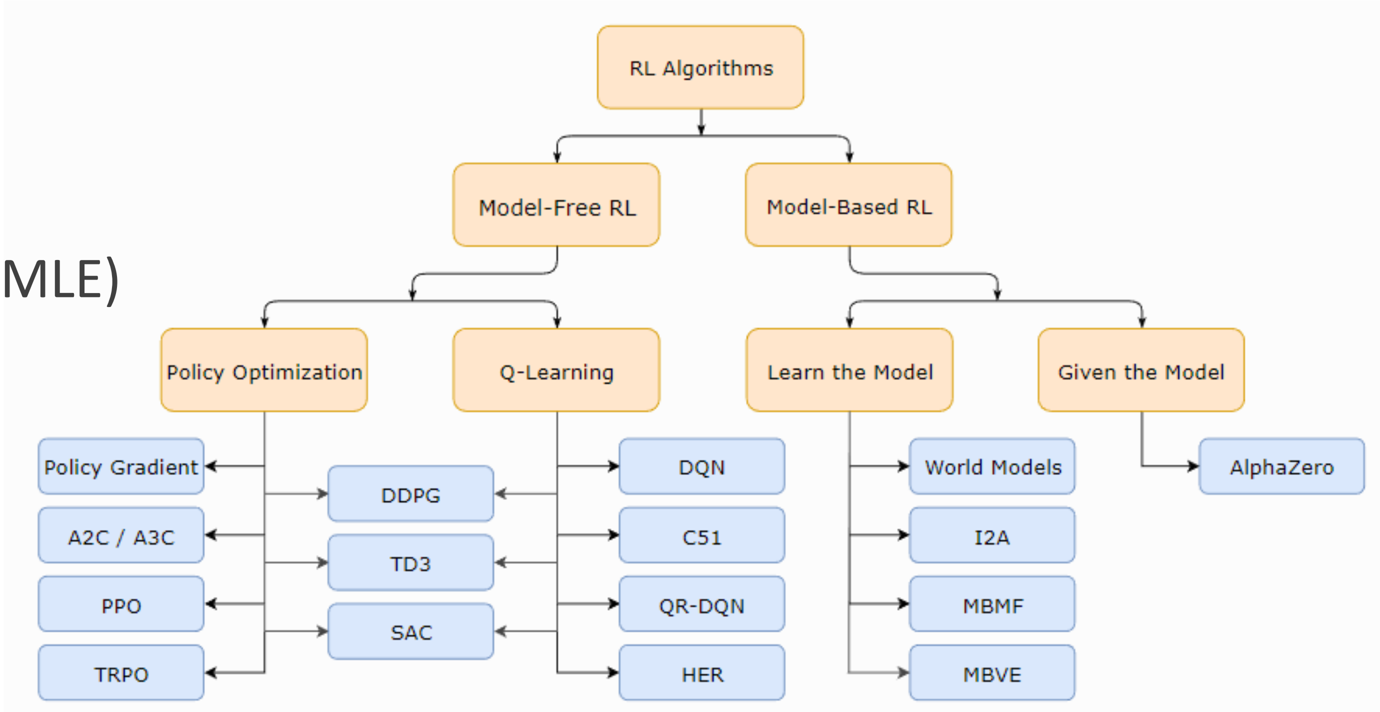


- Given:
  - MDP $\mathcal{M}$ (unknown)
  - The ability to interact with $\mathcal{M}$ for T steps
    - Obtaining trajectory $s_0, a_0, r_0, \dots, s_T, a_T, r_T$
- Goal:
  - (Online learning) maximize cumulative reward $\mathbb{E}[\sum_{t=0}^{T} \gamma^t \, r_t]$
    - Useful in applications where every action taken has real-world consequences (e.g. medical treatment)
  - (Batch learning) output a policy $\hat{\pi}$ such that $V^{\hat{\pi}}$ is competitive with V^*
    - Useful in applications where experimentations are affordable (e.g. laboratory rats, simulators)

Learning in MDPs: A Taxonomy of Approaches

- Model-based RL:

  Repeat:
  - $\widehat{\mathcal{M}} \leftarrow$ Estimate $\mathcal{M}$ based on data (e.g. by MLE)
  - Plan according to $\widehat{\mathcal{M}}$



- Model-free RL: do not estimate $\widehat{\mathcal{M}}$ explicitly
  - Direct policy search
    - E.g. policy gradient (REINFORCE)
  - Value-based methods
    - E.g. Q-learning (this lecture)
  - Actor-critic: combination of the two ideas

https://spinningup.openai.com/en/latest/spinningup/rl_intro2.html

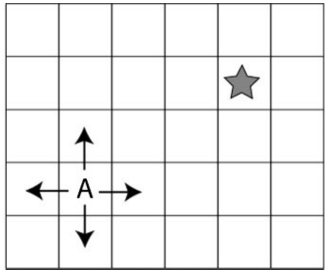
Unique challenges in RL I: Temporal Credit Assignment

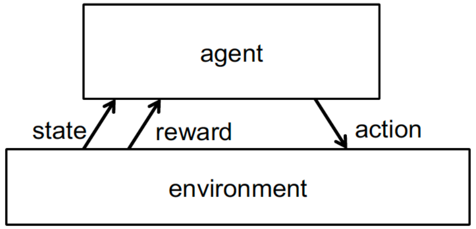
- Performance measure:
  - focuses on the quality of *a sequence of interdependent states / actions*
- Aim for maximization of *long-term rewards*

- E.g.
  - Daily exercise: short term – long term ++
  - Stay up all night playing video games: short term + long term --
  - Chess tactics: sacrifice pieces
- Different from supervised learning: correct classification on every individual examples
- Need to answer questions like: "what is the key step that caused me to lose this game?" – temporal credit assignment

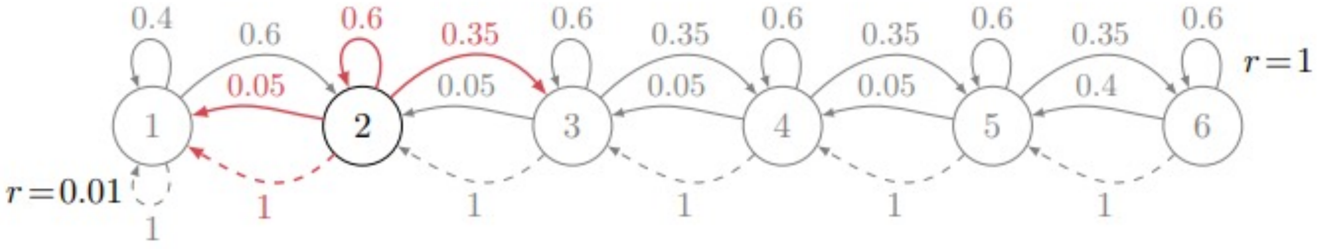
Unique challenges in RL II: Exploration

- Learning agent's data is induced by its own actions
  - This is another key difference with supervised learning
- How to collect *useful* data?
  - The exploration challenge





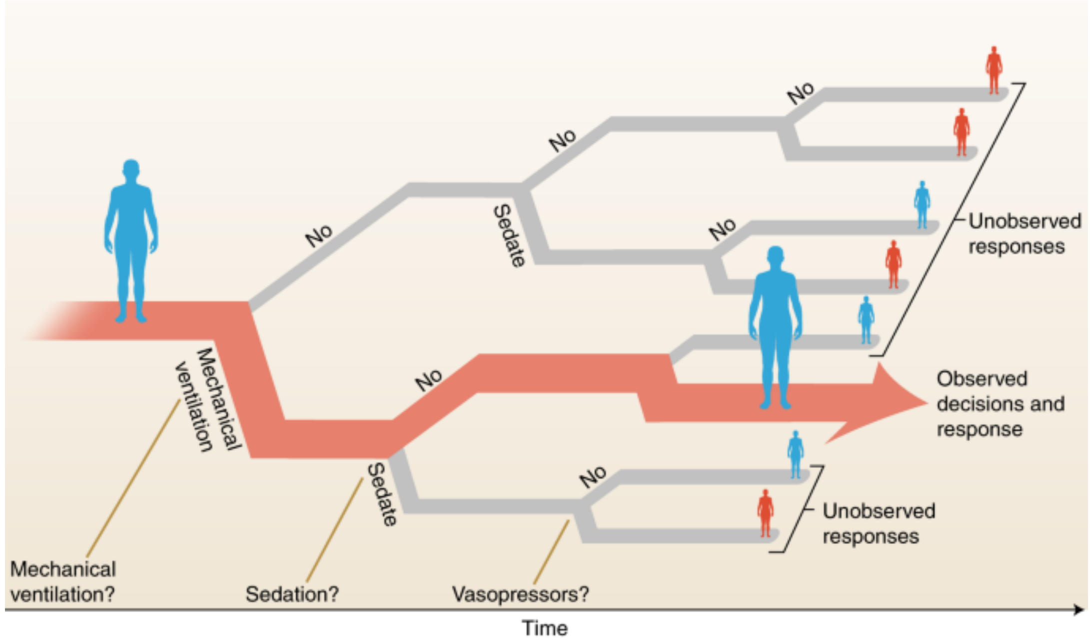
- Rough intuition: collect data that "covers" all states and actions
  - Uniform exploration: take actions uniformly at random
- Caveat: uniform exploration may fail because of some hard-to-reach states
  - E.g. RiverSwim [Strehl & Littman, 2008]



https://rlgammazero.github.io/docs/2020_AAAI_tut_part0.pdf

Unique challenges in RL II: Exploration (cont'd)

- Extra challenge in the *online learning* setting
  - Need to take good actions that yield high rewards
  - Balance *exploration* vs. *exploitation*
  - Not an issue in the batch learning setting



- Popular idea:
  - ϵ-greedy: w.p. $1 - \epsilon$, choose action that is believed to be optimal based on the information collected so far; otherwise, choose actions uniformly at random.
  - Again, ϵ-greedy may fail in some hard MDP environments

https://www.nature.com/articles/s41591-018-0310-5

Monte Carlo Reinforcement Learning

- MC methods learn directly from episodes of experience
- MC is *model-free*: no knowledge of MDP transitions / rewards
- MC learns from complete episodes (no bootstrapping)
- MC uses the simplest idea: value = mean return
- Caveat: Can only apply MC to episodic MDPs (must terminate)

Credit: David Silver

Monte Carlo Reinforcement Learning

Goal: learn V^{π} from episodes of experience under policy π:

$$S_1, A_1, R_2, \ldots, S_k \sim \pi$$

Recall that *return* is total discounted reward:

$$G_t = R_t + \gamma R_{t+1} + \gamma^2 R_{t+2} + \cdots$$

And recall that the *value function* is expected return:

$$V^{\pi}(s) = E_{\pi}[\, G_t \mid S_t = s \,]$$

MC policy evaluation uses *empirical mean* return instead of *expected return*

Credit: David Silver

First-Visit MC Policy Evaluation

- To evaluate *s*
- The **first** time-step t that s is visited in an episode
- Increment counter $N(s) \leftarrow N(s) + 1$
- Increment total return $S(s) \leftarrow S(s) + G_t$
- Estimate value by mean return $V(s) \leftarrow S(s)/N(s)$
- By the law of large numbers $V(s) \rightarrow V^{\pi}$ as $N(s) \rightarrow \infty$

Credit: David Silver

Every-Visit MC Policy Evaluation

- To evaluate *s*
- **Every** time-step t that s is visited in an episode
- Increment counter $N(s) \leftarrow N(s) + 1$
- Increment total return $S(s) \leftarrow S(s) + G_t$
- Estimate value by mean return $V(s) \leftarrow S(s)/N(s)$
- Again, $V(s) \rightarrow V^{\pi}$ as $N(s) \rightarrow \infty$

Credit: David Silver

Example: Blackjack

Objective: Have your card sum be greater than the dealer's without going over 21

States (200 of them)

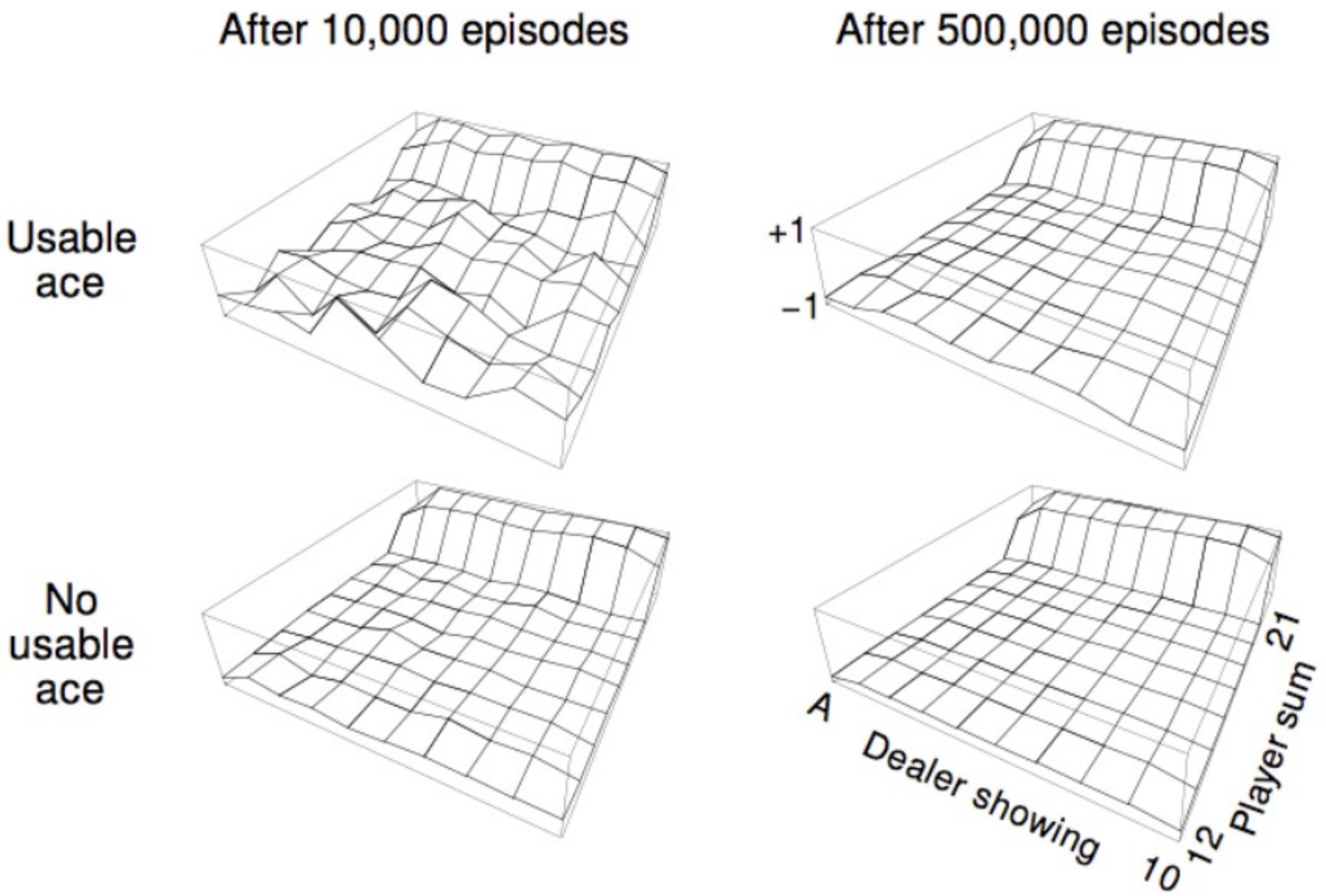
- Current sum (12-21)
- Dealer's showing card (Ace-10)
- Do I have a useable ace?

Reward +1 for winning, 0 for draw, -1 for losing

Actions Hold (stop receiving cards), Hit (receive another card)

Credit: David Silver

Example: Blackjack



Policy Hold if sum at least 20, otherwise hit

Credit: David Silver

Q-functions: motivation

- Issue of V^π: only encodes the quality of states
  - But we need to learn what actions are good
- Is there a function that encodes the quality of actions as well?

Action-value functions (Q-functions):

$$Q^\pi(s,a) = \mathbb{E}[\, G_0 \mid s_0 = s, a_0 = a,\ \pi \,] = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^\pi(s')$$

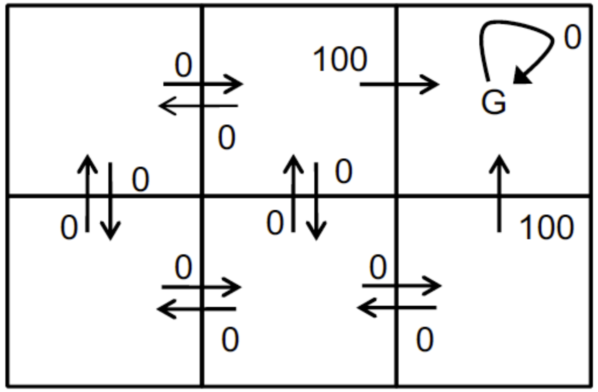
The optimal Q function

$$Q^*(s,a) = \mathbb{E}[\, G_0 \mid s_0 = s, a_0 = a,\ \pi^* \,] = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^*(s')$$

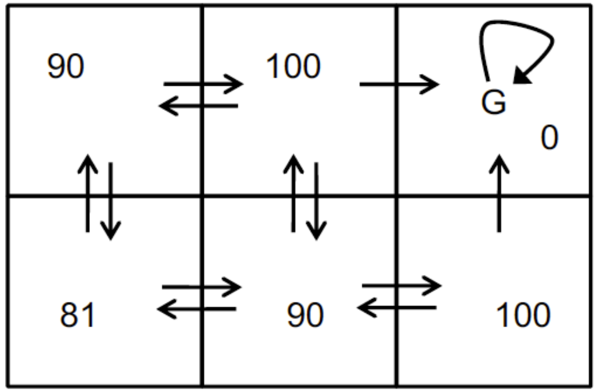
The optimal policy can be extracted from Q^*:

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

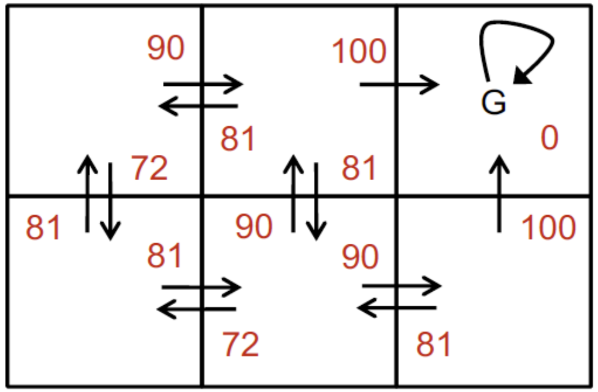
Q-values



$r(s, a)$ (immediate reward) values



$V^*(s)$ values



$Q^*(s, a)$ values

Q-learning: motivation

- We do not know the state transition nor the reward function.

- Instead of learning these model parameters, we directly attempt to estimate Q^*

- Similar to V^*, Q^* also satisfies a Bellman-optimality equation:

$$Q^*(s,a) = R(s,a) + \gamma \cdot \sum_{s'} P(\, s' \mid s, a\,) \max_{a'} Q^*(s', a')$$

Recall: $Q^*(s,a) = r(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^*(s')$

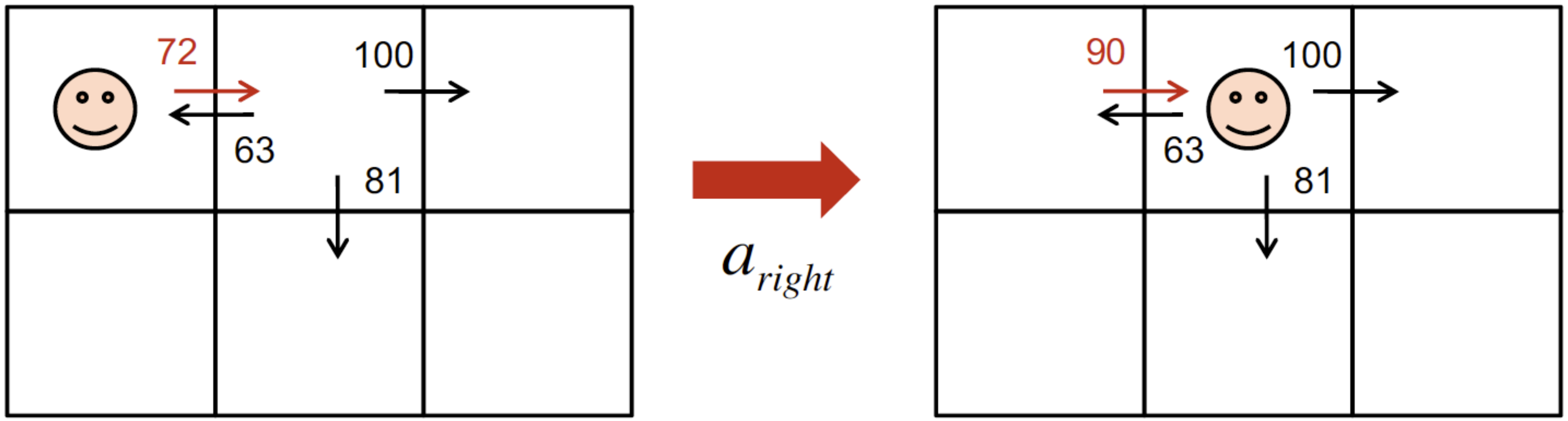
- We will use this to design our learning rule

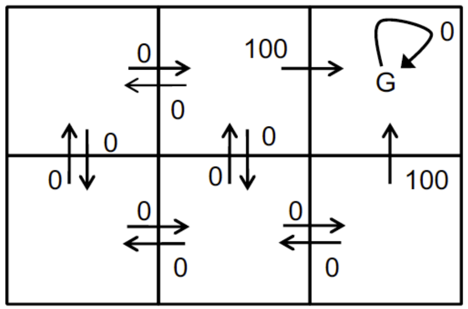
Algorithm: Q-learning (deterministic transitions/rewards)

- Assume that we are in the tabular setting: S and A are both finite
- Initialize: $Q(s, a) = 0, \forall s, a$
- Observe the initial state s
- Repeat:
  - Select an action a and execute it (e.g., ϵ-greedy)
  - Receive a reward r
  - Observe a new state s'
  - Update: $Q(s, a) \leftarrow r + \gamma \max_{a'} Q(s', a')$ (similar to value iteration)
  - $\mathrm{s} \leftarrow s'$

$$\boxed{Q^*(s, a) = R(s, a) + \gamma \cdot \sum_{s'} P(\, s' \mid s, a\,) \max_{a'} Q^*(s', a')}$$

Q-learning: update example





$r(s, a)$ (immediate reward) values

$$Q(s_1, a_{right}) \leftarrow r + \gamma \max_{a'} Q(s_2, a')$$
$$\leftarrow 0 + 0.9 \max\{63, 81, 100\}$$
$$\leftarrow 90$$

Q-learning for stochastic transitions/rewards

- Our update equation is problematic: $Q(s,a) \leftarrow r + \gamma \max_{a'} Q(s',a')$

$$s_0 \xrightarrow[r_0]{a_0} s_1 \xrightarrow[r_1]{a_1} s_2 \xrightarrow[r_2]{a_2}$$

- For stochastic worlds:
  - Fix s, a, (next state, reward) s', r seen is stochastic
  - Even if $Q = Q^*$ in the previous iteration, $Q(s,a)$ will deviate from $Q^*(s,a)$ after the update
  - This results in $Q(s,a)$ not converging

- How to fix this? Recall:

$$\boxed{Q^*(s,a) = R(s,a) + \gamma \cdot \sum_{s'} P(s' \mid s,a) \max_{a'} Q^*(s',a')}$$

- We can use the idea of stochastic approximation (also called temporal difference learning in the RL context)

Stochastic approximation

- Given a *stream* of data points $X_1, \ldots, X_n \sim N(\mu, 1)$
- How to estimate μ in an *anytime* manner?

- Idea 1: at time step n, output estimate $\hat{\mu}_n = X_n$

- Can we do better?

- Idea 2: at time step n, output estimate $\hat{\mu}_n = \frac{1}{n}(X_1 + \cdots + X_n)$
- This is equivalent to $\hat{\mu}_n = \underbrace{(1 - \alpha_n)\hat{\mu}_{n-1}}_{\substack{\text{Old estimate} \\ \text{(conservativenss)}}} + \underbrace{\alpha_n X_n}_{\substack{\text{New data} \\ \text{(correctivenss)}}}$, where $\alpha_n = \frac{1}{n}$

Q-learning for nondeterminstic transitions/rewards

- Initialize: $Q(s, a) = 0, \forall s, a$
- Observe the initial state s
- Repeat
  - Take an action a
    - e.g., ϵ-greedy (taking $\text{argmax}_a Q(s, a)$ w.p. $1 - \epsilon$)
  - Receive the reward r
  - Observe the new state s’
  - Update: $Q(s, a) \leftarrow (1 - \alpha) Q(s, a) + \alpha \left(r + \gamma \max_{a'} Q(s', a') \right)$
  - $s \leftarrow s'$

$$\boxed{Q^*(s, a) = R(s, a) + \gamma \cdot \sum_{s'} P(s' \mid s, a) \max_{a'} Q^*(s', a')}$$

α is a hyperparameter! (next slide)

The choice of α

- $Q(s,a) \leftarrow (1-\alpha)Q(s,a) + \alpha\left(r + \gamma \max_{a'} Q(s',a')\right)$
- For example, $\alpha = \frac{1}{1 + \#\mathrm{times}(s,a)}$.
- Q: Why is this a reasonable choice?

Discussion

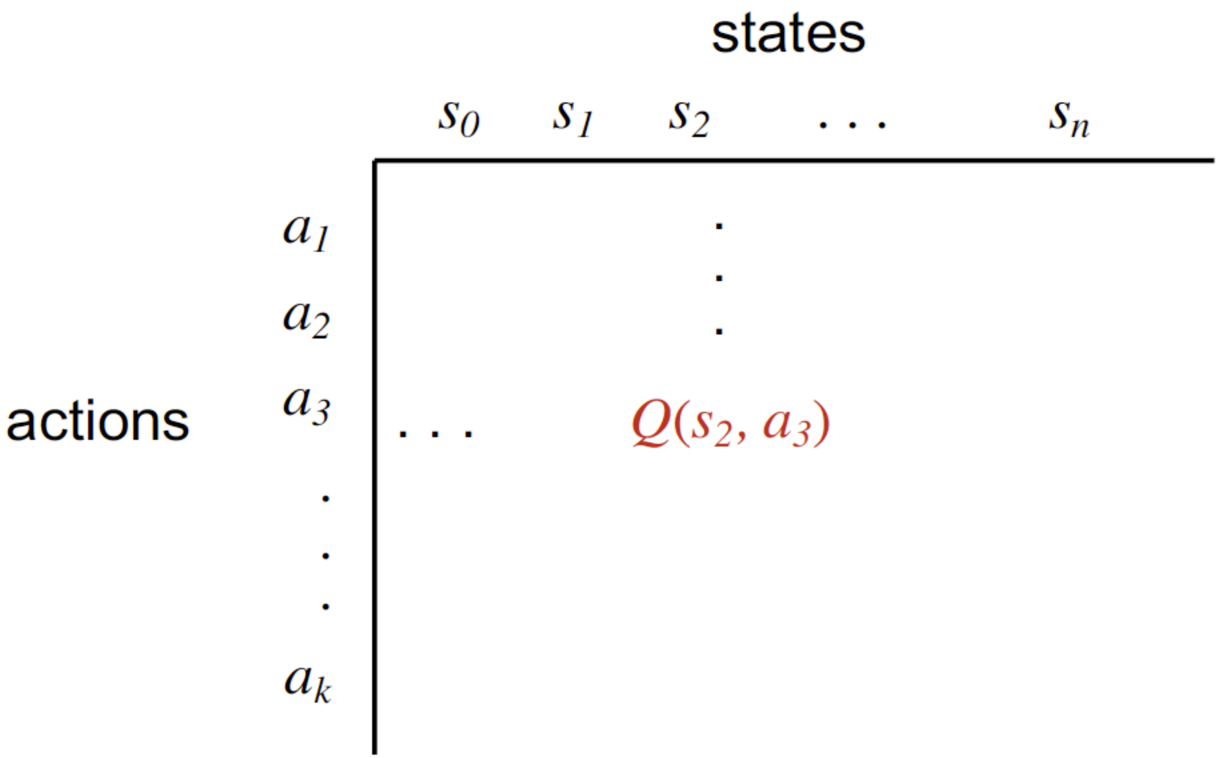
- Q-learning will converge to the optimal Q function (under certain niceness assumptions on the MDP, exploration policy, and step size scheme)

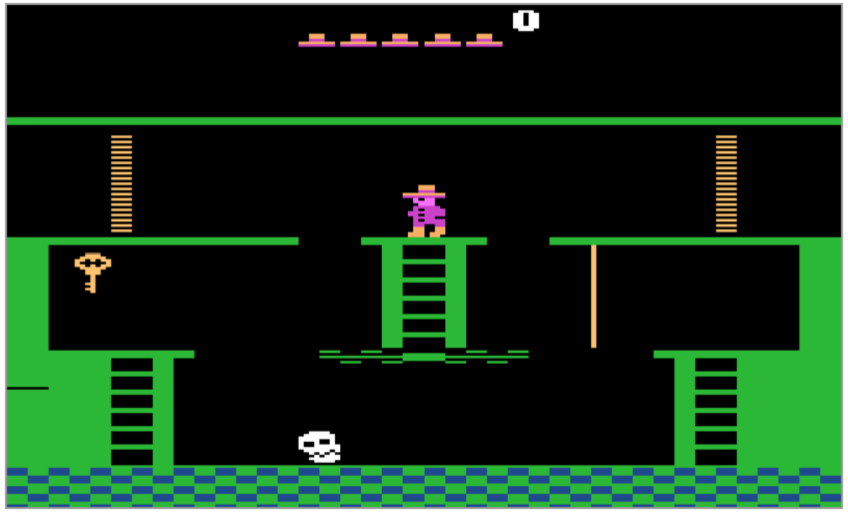
- In practice, it takes a lot of iterations!

- Comparison: Model-based learning vs. Q-learning when choosing actions
  - Model-based
    - need to look ahead using some estimates of rewards and transition probabilities (Model Predictive Control)
  - Q-learning (model-free)
    - just choose the action with the largest Q value

Challenge of Q-learning: large state spaces

- Q-learning requires us to maintain a huge table, which is clearly infeasible with large state spaces





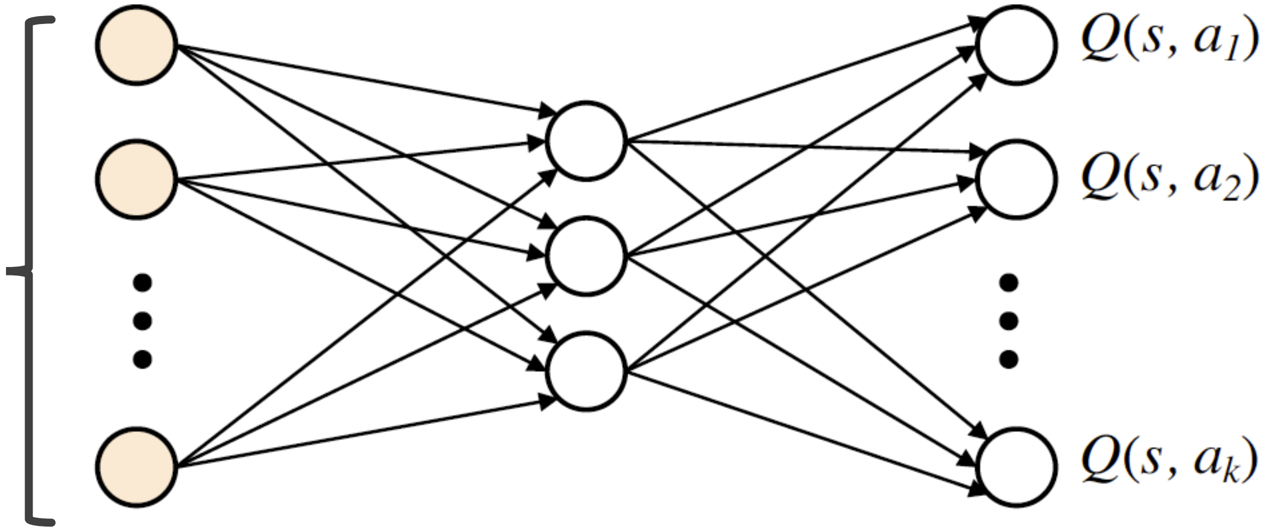
- How to design a Q-learning-style algorithm that can handle large state spaces?

https://www.microsoft.com/en-us/research/uploads/prod/2018/09/Reinforcement-Learning-with-Rich-Observations-SLIDES.pdf

Q function approximation

- We can use some other function representation (e.g. a neural net) to compactly encode a substitute for the big table.
- We've been thinking states as discrete (the set S), but in fact, they can be a feature vector!

encoding of the state



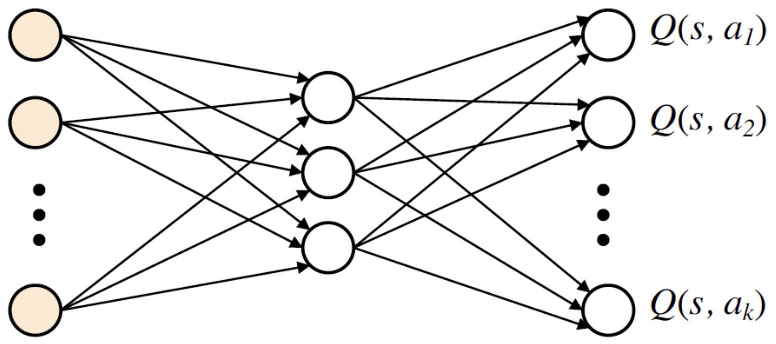
each input unit can be a sensor value
(or more generally, a feature)

Q: why is this a good idea?

Why Q function approximation?

- 1. memory issue
- 2. is able to *generalize across states*! may speed up the convergence.

- Example: 100 binary features for states. 10 possible actions.
- Q table size = 10 x 2^{100} entries
- NN with 100 hidden units:
  - 100 x 100 + 100 x 10 = 11k weights (not counting bias for simplicity)



Algorithm: fitted Q-learning

Repeat

- observe the state s
- compute $Q(s, a)$ for each action a (forward pass on the NN)
- select action a (e.g. use ϵ-greedy) and execute it
- observe the new state s' and the reward r
- compute $Q(s', a')$ for each action a' (forward pass on the NN)
- update the NN with the instance
  - $x \leftarrow s$
  - $y \leftarrow (1-\alpha)Q(s,a) + \alpha\left(r + \gamma \cdot \max_{a'} Q(s', a')\right)$ (label for Q(s,a))

Calculate Q value you would have put into the Q-table and use it as the training label.
Use the squared loss and perform backpropagation!

Fitted Q-learning example: Atari games

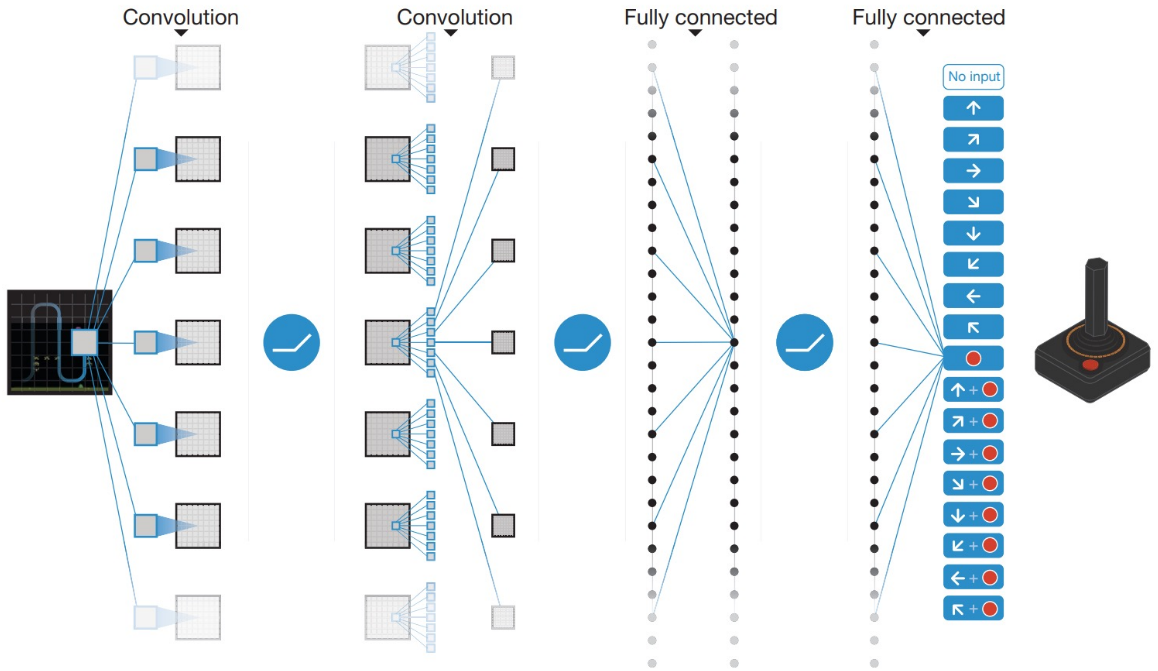
- Human-level control through deep reinforcement learning (Mnih et al, 2013, 2015)
- Tested Fitted Q-learning on 49 Atari games

- Achieves >=75% of human professional players' scores on 29 games
- Can significantly outperform human players in many games

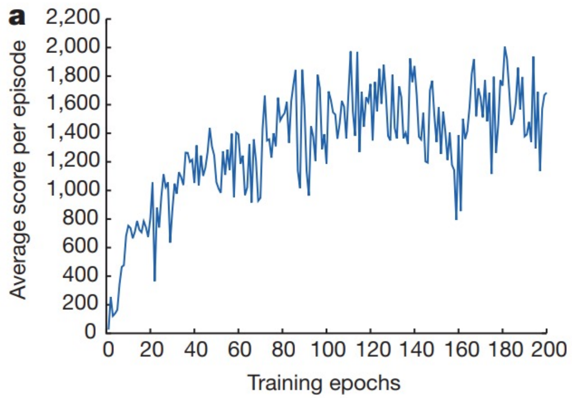
https://arxiv.org/pdf/1312.5602.pdf
https://www.nature.com/articles/nature14236

Fitted Q-learning example: Atari games (cont'd)

- The neural network for fitting Q values
  - Convolutional architecture to handle states as images

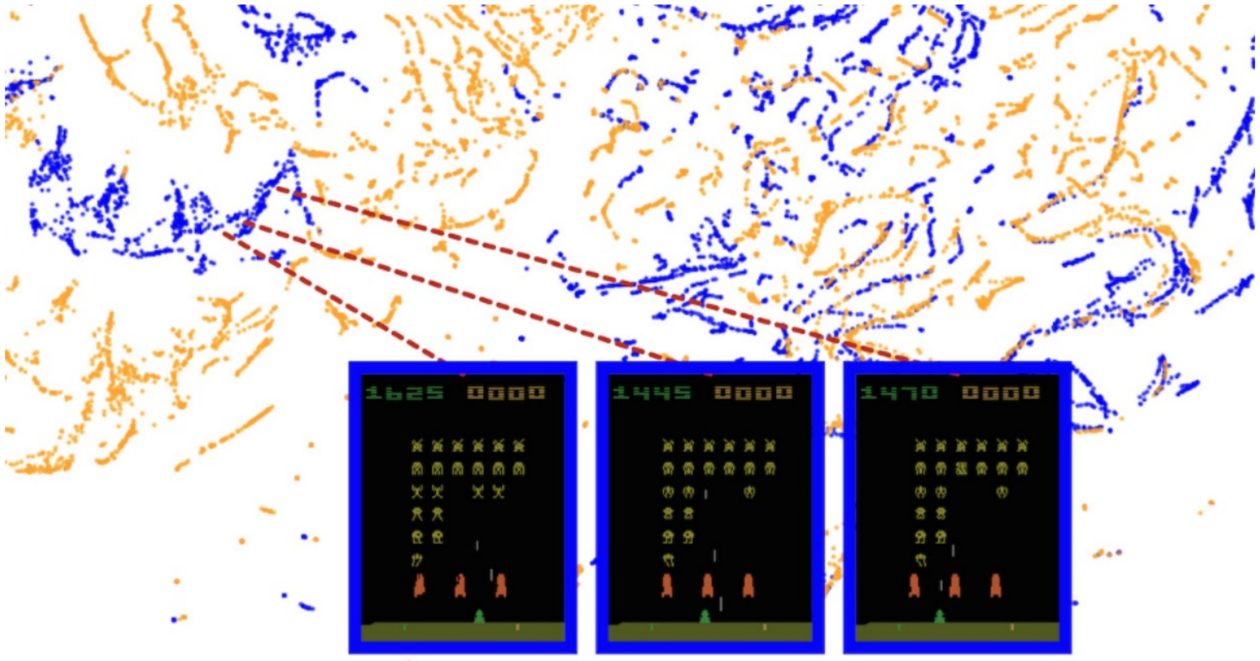


- Learning curve: (Space Invaders, ϵ-greedy with $\epsilon = 0.05$)



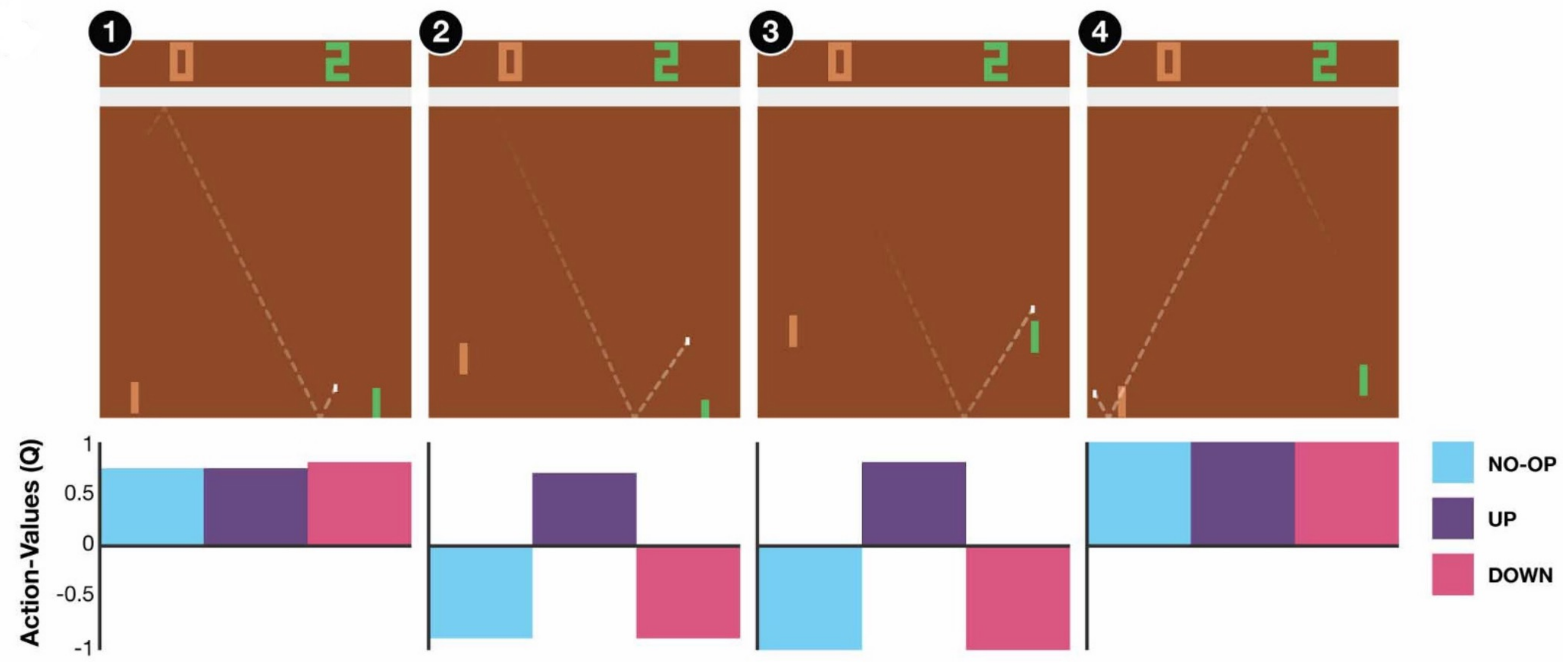
Fitted Q-learning example: Atari games (cont’d)

- Q-network’s last hidden layer extracts useful representations
- Consequently Q-network provides Q-value estimates that generalize across states



Fitted Q-learning example: Atari games (cont'd)

- The learned Q functions are sensible



Summary

- MDPs: Reward driven philosophy
- Policy evaluation: Bellman consistency equations; fixed point iteration
- Planning in MDPs: value iteration; policy iteration
- Learning in MDPs: Q-learning; function approximation