Computer Science

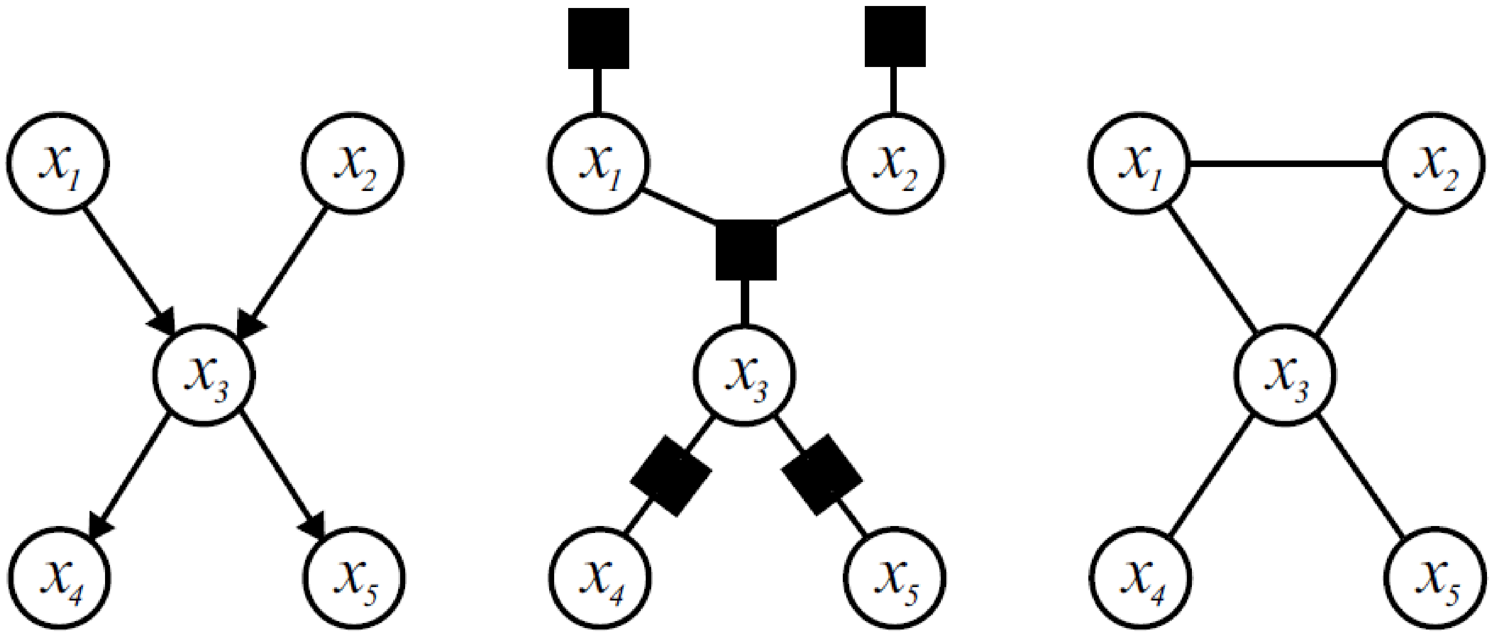
CSC535: Probabilistic Graphical Models

Probabilistic Graphical Models

Prof. Jason Pacheco

Graphical Models

A variety of graphical models can represent the same probability distribution



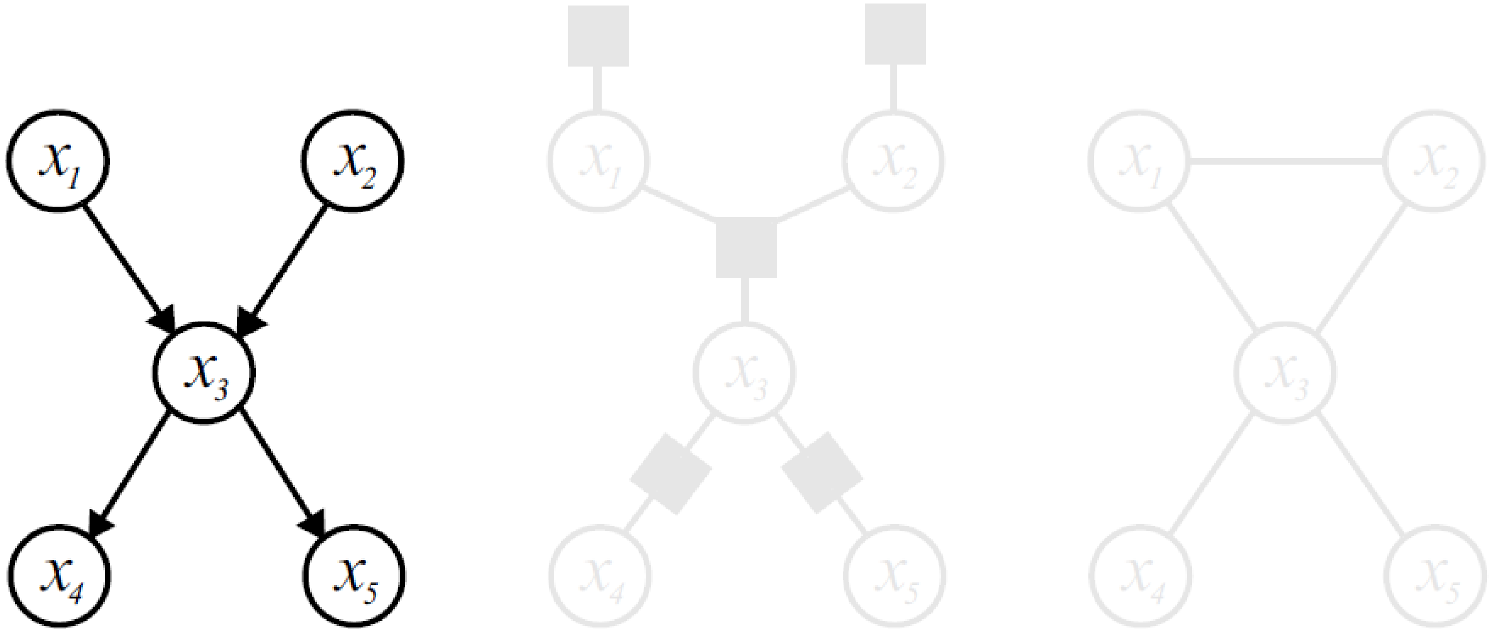
Bayes Network **Factor Graph** **Markov Random Field**

Directed Models **Undirected Models**

[Source: Erik Sudderth, PhD Thesis]

Graphical Models

A variety of graphical models can represent the same probability distribution



Bayes Network | **Factor Graph** | **Markov Random Field**

Directed Models | **Undirected Models**

[Source: Erik Sudderth, PhD Thesis]

Outline

Directed graphical models

- Bayes Nets
- Conditional dependence

Undirected graphical models

- Markov random fields (MRFs)
- Factor graphs

Outline

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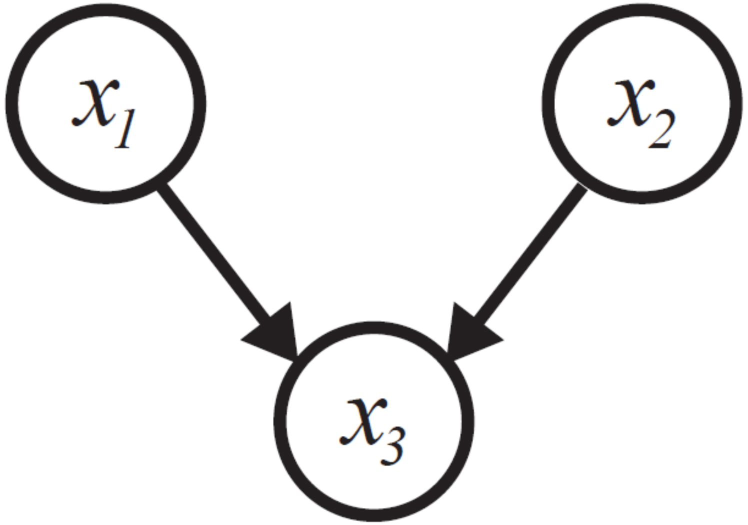
From Probabilities to Pictures

A probabilistic graphical model allows us to pictorially represent a probability distribution

Probability Model:

$$p(x_1, x_2, x_3) = p(x_1)p(x_2)p(x_3 \mid x_1, x_2)$$

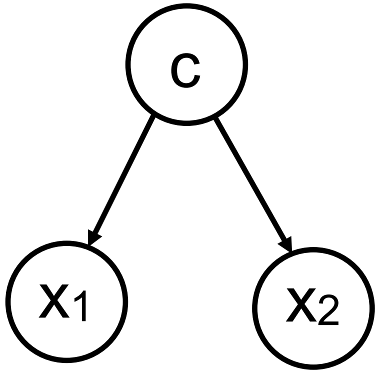
Graphical Model:



Conditional distribution on each RV is dependent on its parent nodes in the graph

Directed Graphical Models

Directed models are generative models…



$$p(C, X_1, X_2) = p(C)p(X_1 \mid C)p(X_2 \mid C)$$

The graph and the formula say exactly the same thing.
(The graph has very specific semantics.)

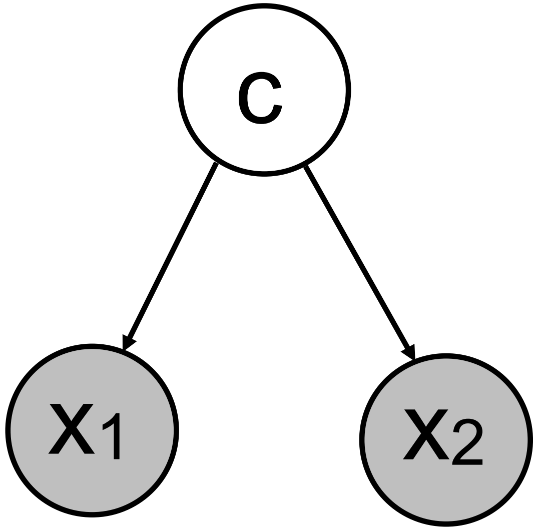
…tells how data are generated (called ***ancestral sampling***)

Step 1 Sample root node (prior): $c \sim p(C)$

Step 2 Sample children, given sample of parent (likelihood):

$$x_1 \sim p(X_1 \mid C = c) \qquad x_2 \sim p(X_2 \mid C = c)$$

Inference



Denote observed data with shaded nodes,

$$X_1 = x_1 \qquad\qquad X_2 = x_2$$

Infer *latent* variable C via Bayes' rule:

$$p(c \mid x_1, x_2) = \frac{p(c)p(x_1 \mid c)p(x_2 \mid c)}{p(x_1, x_2)}$$

- This is (obviously) a simple example
- Models and inference task can get really complicated
- But the fundamental concepts and approach are the same

Chain Rule of Probability

Recall the **probability chain rule** says that we can decompose any joint distribution as a product of conditionals….

$$p(x_1, x_2, x_3, x_4) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1, x_2)p(x_4 \mid x_1, x_2, x_3)$$

Valid for *any ordering* of the random variables…

$$p(x_1, x_2, x_3, x_4) = p(x_3)p(x_1 \mid x_3)p(x_4 \mid x_1, x_3)p(x_2 \mid x_1, x_3, x_4)$$

For a collection of N RVs and any permutation ρ:

$$p(x_1, \dots, x_N) = p(x_{\rho(1)}) \prod_{i=2}^{N} p(x_{\rho(i)} \mid x_{\rho(i-1)}, \dots, x_{\rho(1)})$$

Conditional Independence

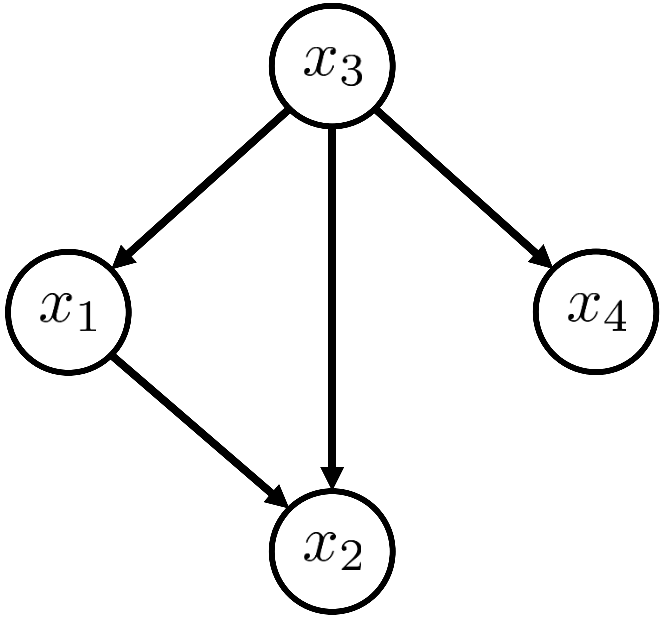
Recall two RVs X and Y are **conditionally independent** given Z (or $X \perp Y \mid Z$) iff:

$$p(X \mid Y, Z) = p(X \mid Z)$$

Idea Apply *chain rule* with ordering that exploits conditional independencies to simplify the terms

Ex. Suppose $x_4 \perp x_1 \mid x_3$ and $x_2 \perp x_4 \mid x_1$ then:

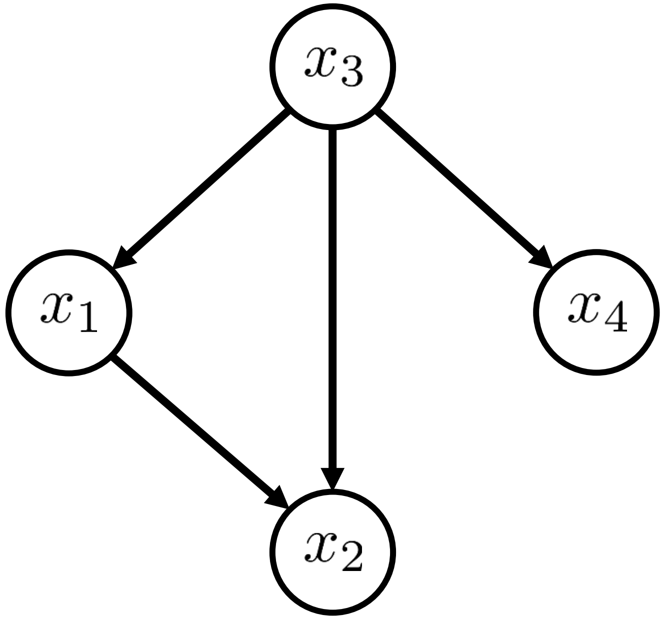
$$\begin{aligned} p(x) &= p(x_3)p(x_1 \mid x_3)p(x_4 \mid x_1, x_3)p(x_2 \mid x_1, x_3, x_4) \\ &= p(x_3)p(x_1 \mid x_3)p(x_4 \mid x_3)p(x_2 \mid x_1, x_3) \end{aligned}$$



Can visualize conditional dependencies using ***directed acyclic graph*** *(DAG)*

General Directed Graphs

Def. A <u>directed graph</u> is a graph with edges $(s,t) \in \mathcal{E}$ (arcs) connecting parent vertex $s \in \mathcal{V}$ to a child vertex $t \in \mathcal{V}$



Def. <u>Parents</u> of vertex $t \in \mathcal{V}$ are given by the set of nodes with arcs pointing to t,

$$\mathrm{Pa}(t) = \{s : (s,t) \in \mathcal{E}\}$$

<u>Children</u> of $t \in \mathcal{V}$ are given by the set,

$$\mathrm{Ch}(t) = \{t : (t,k) \in \mathcal{E}\}$$

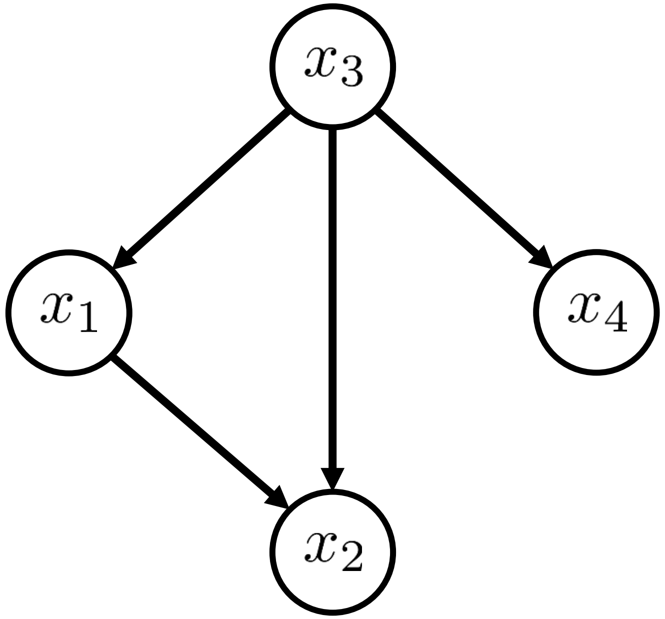
<u>Ancestors</u> are parents-of-parents.
<u>Descendants</u> are children-of-children.

Directed PGM = Bayes Network

Model factors are normalized conditional distributions:

$$p(x) = \prod_{s \in \mathcal{V}} p(x_s \mid x_{\mathrm{Pa}(s)})$$

Parents of node *s*



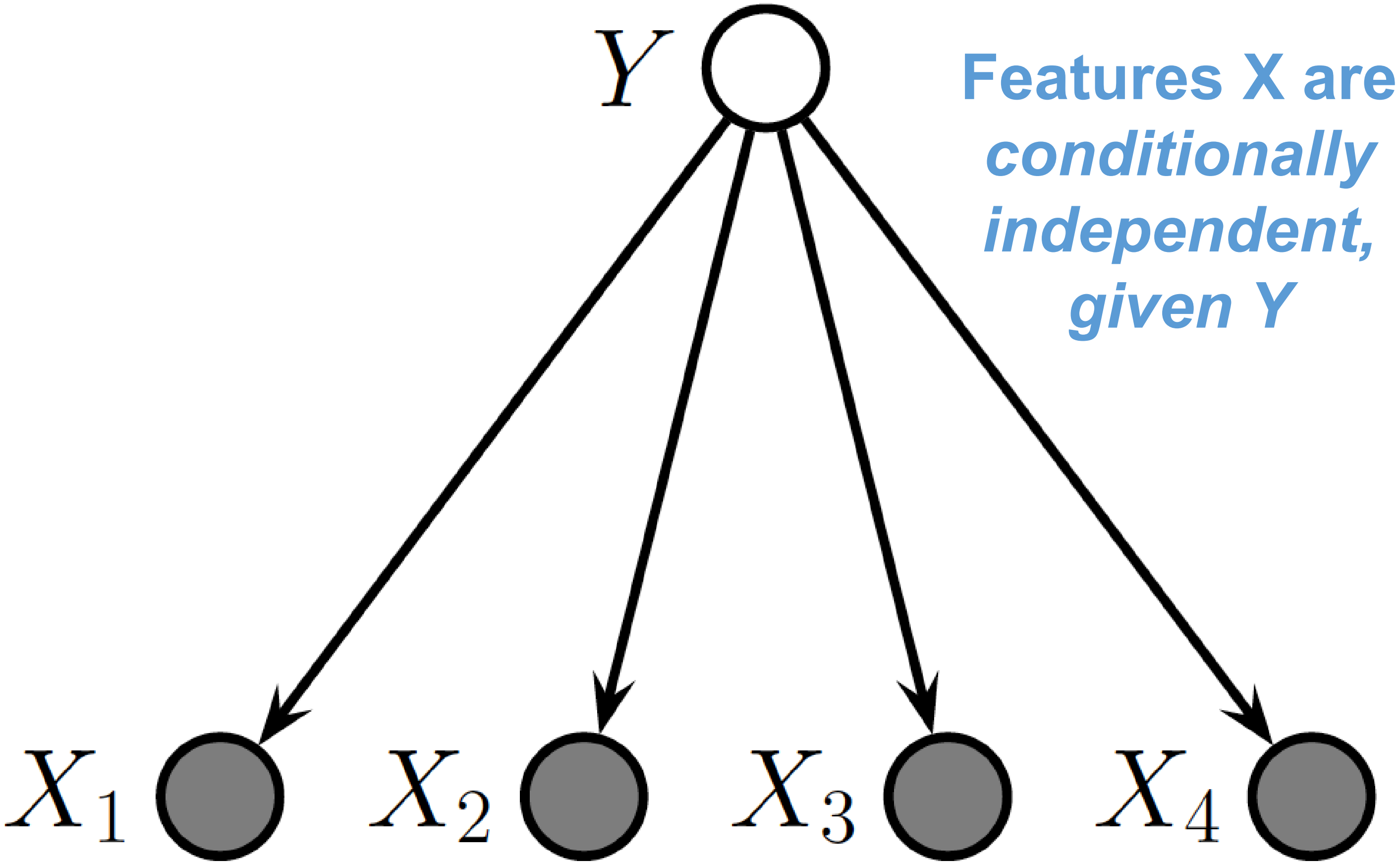
Directed acyclic graph (DAG) specifies factorized form of joint probability:

$$p(x) = p(x_3)p(x_1 \mid x_3)p(x_4 \mid x_3)p(x_2 \mid x_1, x_3)$$

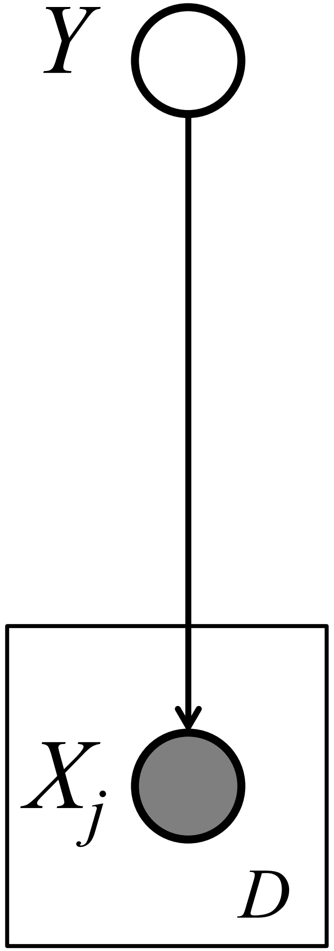
Locally normalized factors yield *globally normalized* joint probability

Shading & Plate Notation

Convention: *Shaded nodes are observed, open nodes are latent/hidden/unobserved*



Features X are *conditionally independent, given Y*



Plates denote replication of random variables

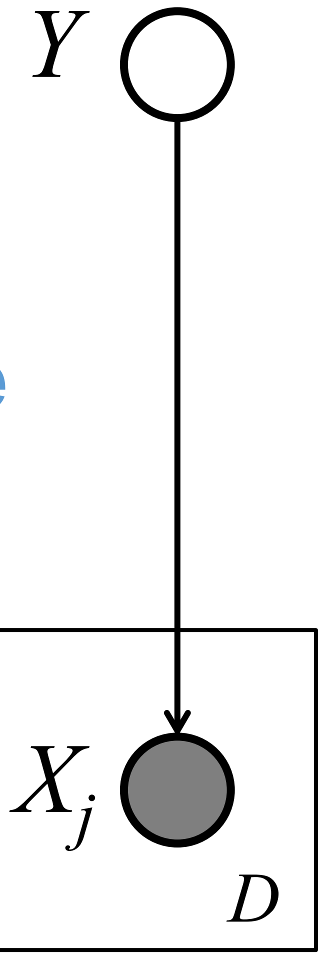
$$p(y, \mathbf{x}) = p(y) \prod_{j=1}^{D} p(x_j|y)$$

Question Does anybody know the name for this model? **Naïve Bayes**

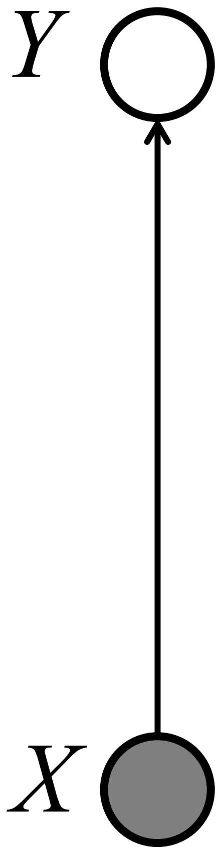
Inference

Interpret inference as inverting arrows in the graphical model

Naïve Bayes Generative Model



$$p(y, \mathbf{x}) = p(y) \prod_{j=1}^{D} p(x_j | y)$$



Posterior Model

Posterior

Marginal Likelihood

$$p(y, \mathbf{x}) = p(y \mid \mathbf{x}) p(\mathbf{x})$$

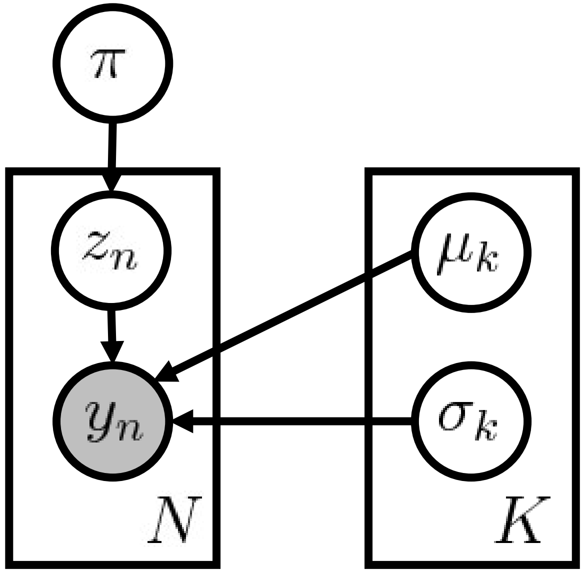
Example: Gaussian Mixture Model

Bayes nets are easily simulated via ancestral sampling…

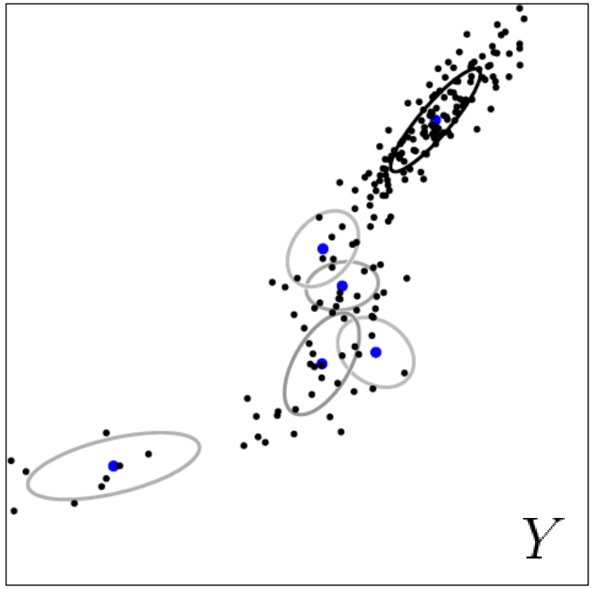
Probability Model

$$\pi \sim \mathrm{Dirichlet}(\cdot)$$
$$\mu_k \sim \mathcal{N}(\cdot)$$
$$\sigma_k \sim \text{Inv-Gamma}(\cdot)$$
$$z_n \mid \pi \sim \mathrm{Cat}(\pi)$$
$$y_n \mid z_n, \mu_{z_n}, \sigma_{z_n} \sim \mathcal{N}(\mu_{z_n}, \sigma_{z_n})$$

Bayes Net

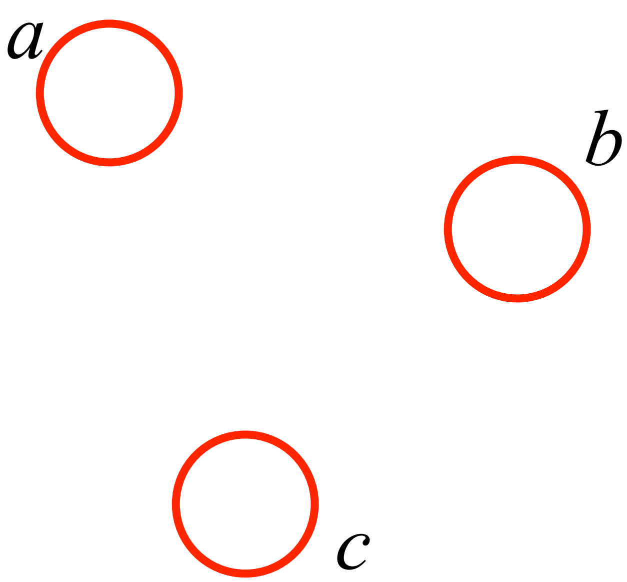


Joint Sample

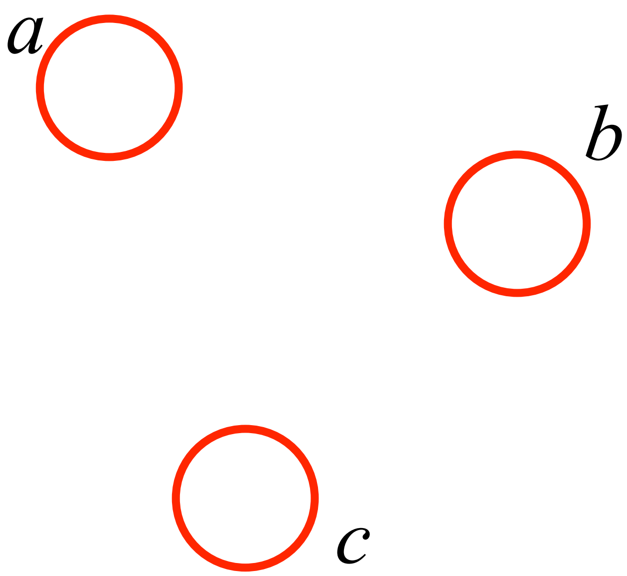


Sample all nodes with no parents, then children, etc., to terminals. Can sample nodes at same level in parallel.

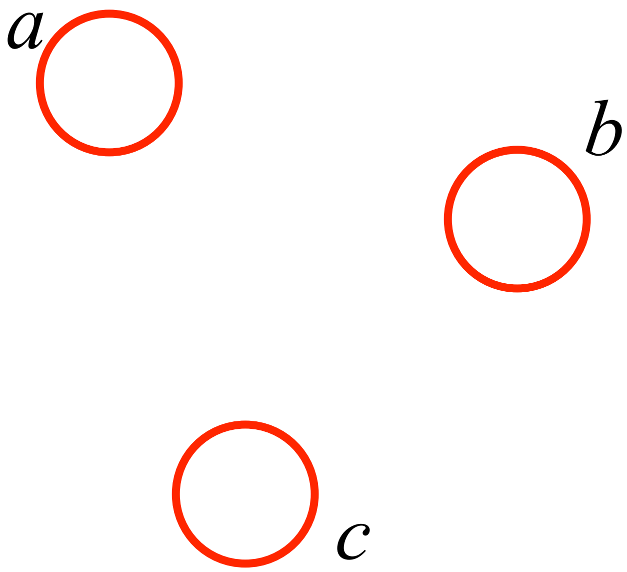
What is the joint factorization?



$$p(a,b,c) = p(a)p(b)p(c)$$

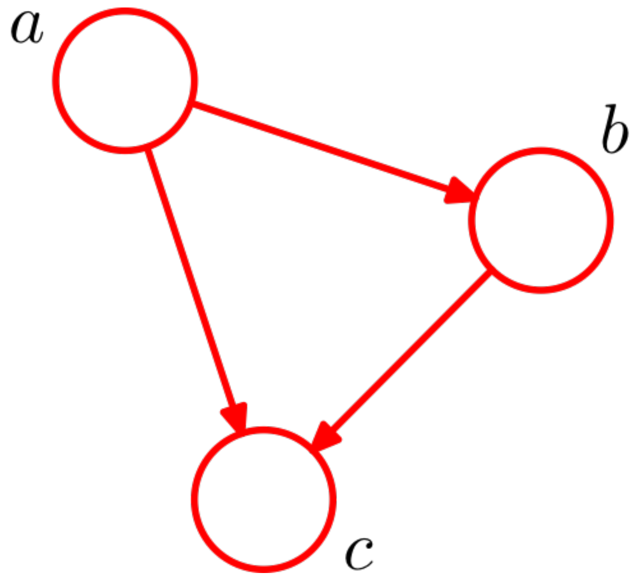


Are a and b independent ($a \perp b$)?



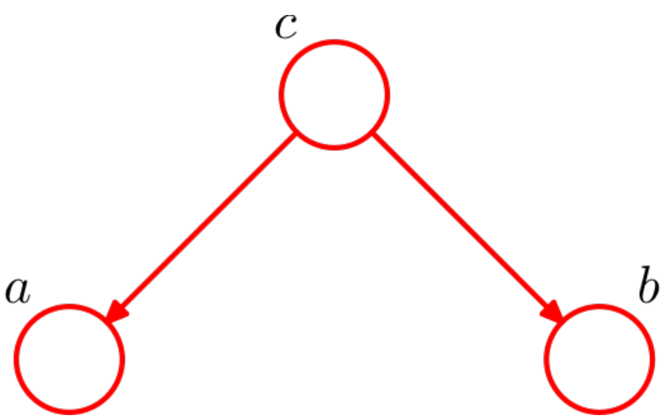
$$p(a,b,c) = p(a)p(b)p(c)$$

$$p(a,b,c) = p(a)p(b|a)p(c|a,b)$$

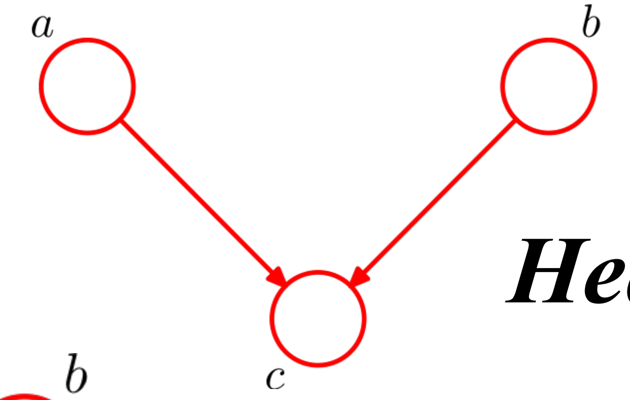


Note there are **no conditional independencies**

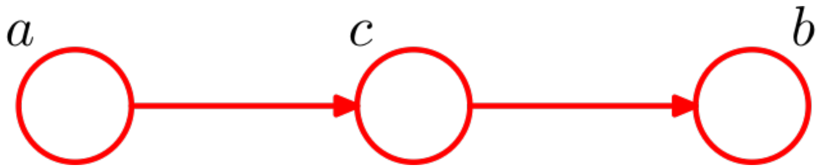
Three interesting cases



Tail-to-tail

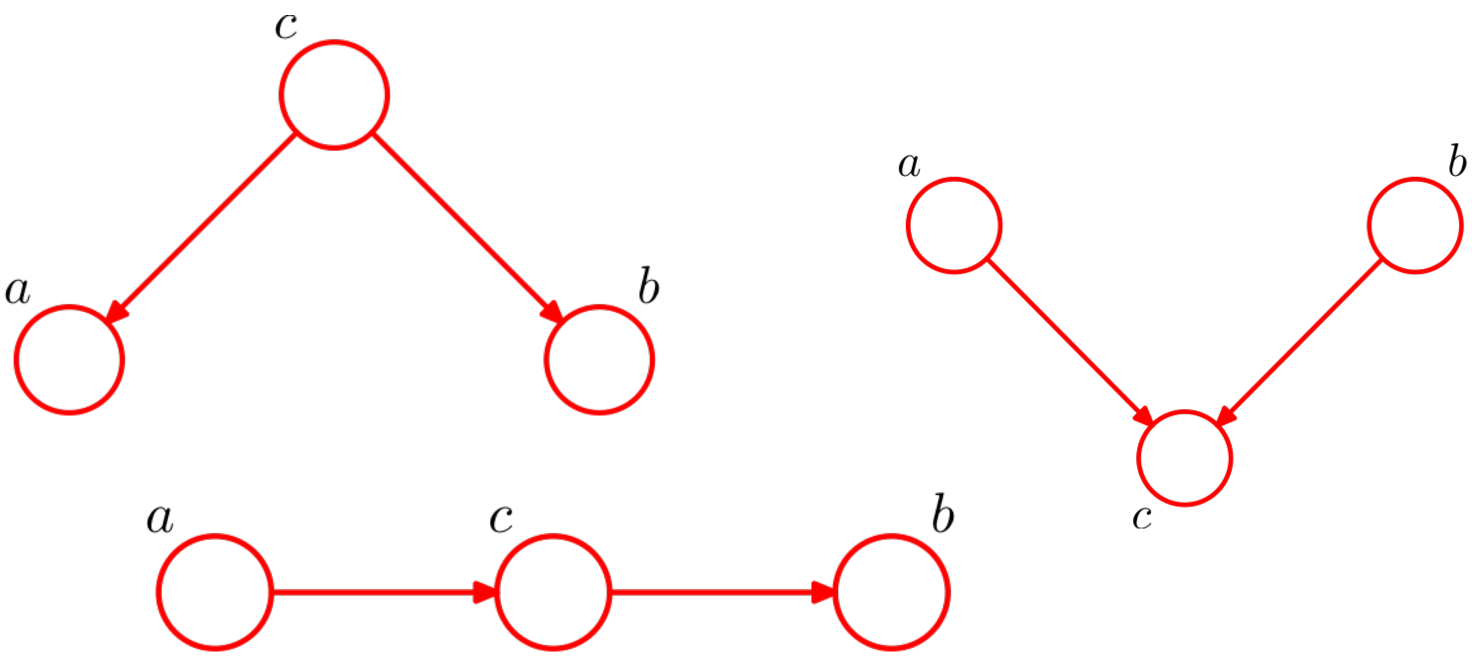


Head-to-head



Head-to-tail

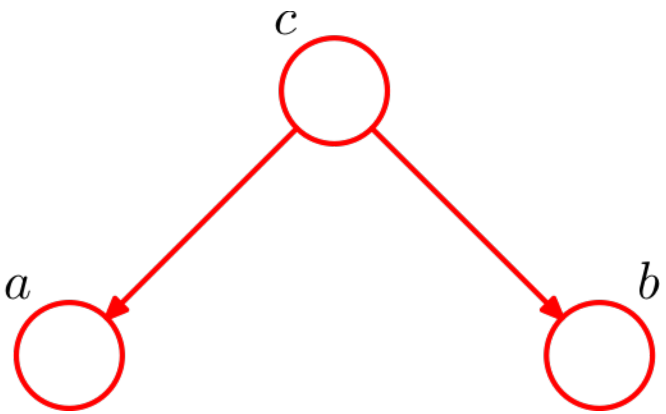
Three interesting cases



For each case, consider two questions:

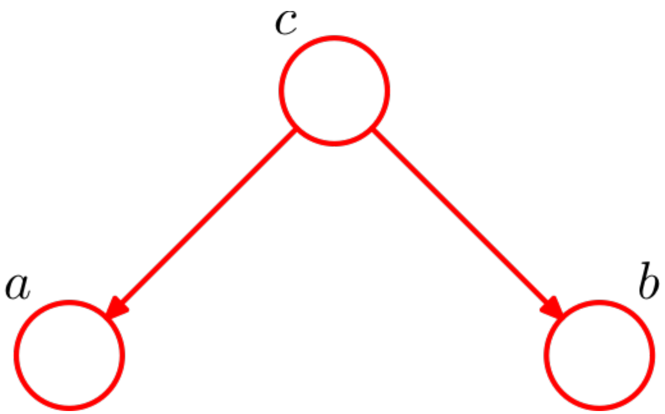
1) Is $a \perp b$?
2) Is $a \perp b \mid c$? (i.e. c is observed)

Case one (tail-to-tail)



Is $a \perp b$?

Case one (tail-to-tail)

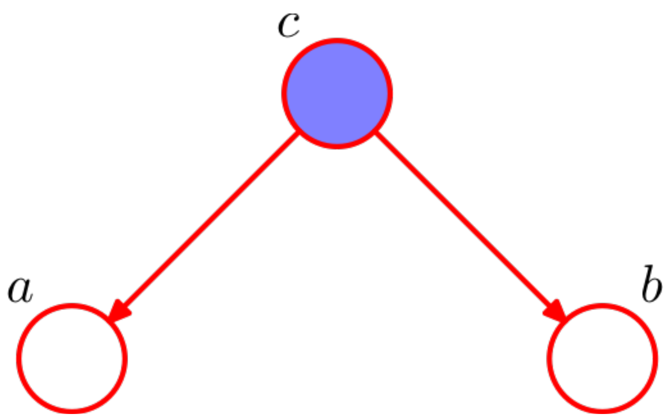


$$a \not\perp\!\!\!\perp b$$

Intuition *c* generates, both, *a* and *b*. Knowing *a* tells you something about *c* (via Bayes rule $p(c|a)$) which in turn generates b…information is exchanged

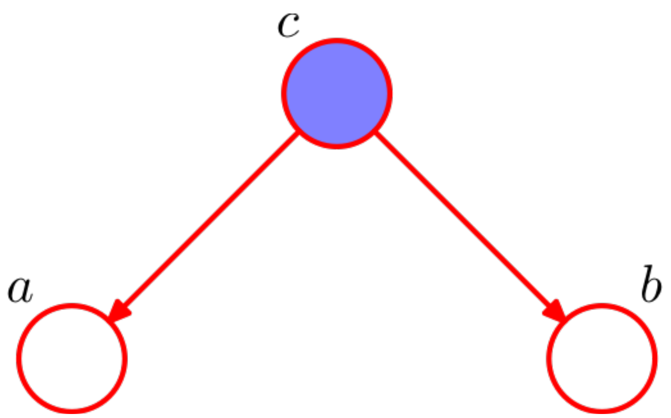
Can prove by counterexample (HW problem)

Case one where c is observed



Is $a \perp b \mid c$?

Case one where c is observed



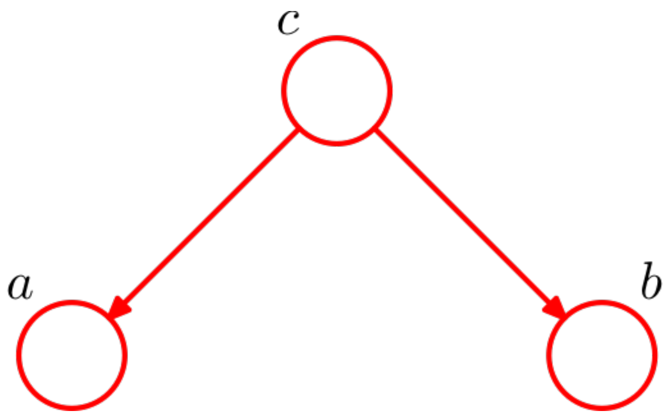
$$a \perp b \mid c$$

$$p(a,b,c) = p(c)p(a|c)p(b|c) \quad \text{(what the graph represents in general)}$$

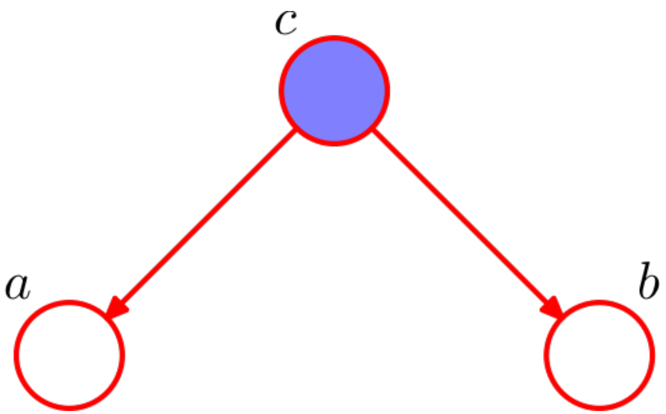
$$p(a,b|c) = p(a|c)p(b|c) \quad \text{(with } c \text{ observed)}$$

This is the definition of $a \perp b | c$

Case one (tail-to-tail) summary



$$a \not\perp b$$



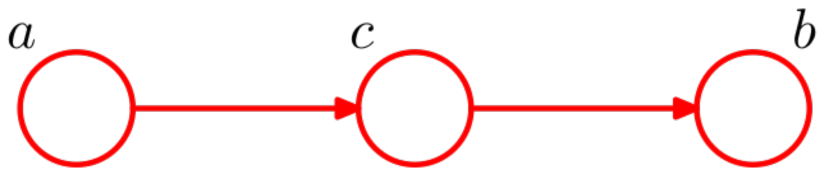
$$a \perp b \mid c$$

Tail-to-tail case

With no conditioning = no independence

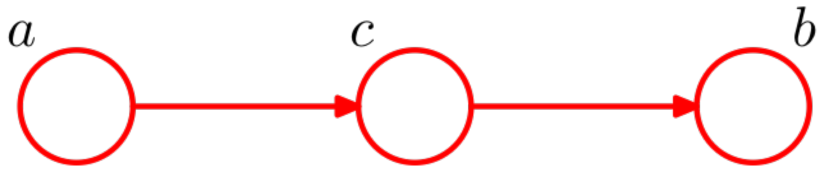
With conditioning = independence

Case two (head-to-tail)



Is $a \perp b$?

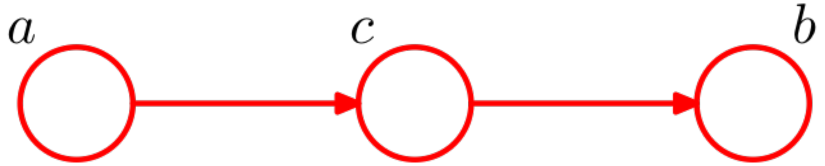
Case two (head-to-tail)



$$a \not\perp\!\!\!\perp b$$

If you know *a*, that informs you about *c*, which informs you about *b*.

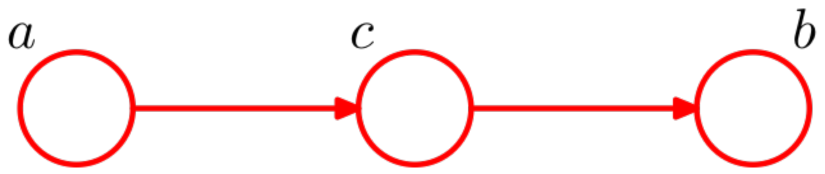
Case two (head-to-tail)



$$a \not\perp b$$

The graph represents $p(a,b,c) = p(a)\,p(c|a)\,p(b|c)$

Case two (head-to-tail)



$$a \not\perp b$$

The graph represents $p(a,b,c) = p(a)\,p(c|a)\,p(b|c)$

Algebraically,

$$p(a,b) = \sum_c p(a,b,c) = p(a)\sum_c p(c|a)\,p(b|c)$$

If $a \perp b$ then the above would also have to be equal to $p(a)p(b)$

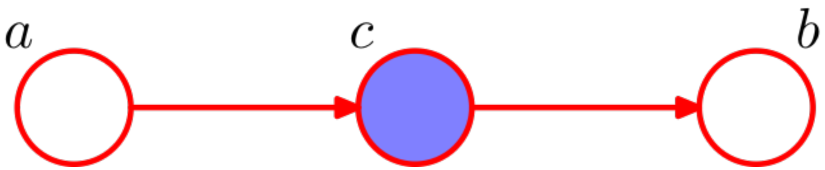
$$p(a,b) = \sum_c p(a,b,c) = p(a) \sum_c p(c|a)\, p(b|c)$$

If $a \perp b$ then the above **also** equals $p(a)p(b)$

To prove the claim that a $\not\perp$ b we can construct a counter example where the above is false.

Homework Question

Case two where c is observed

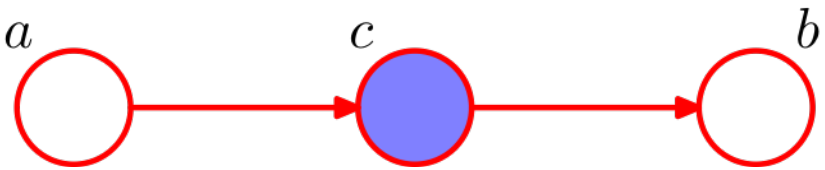


$$a \perp b \mid c$$

$$p\left(a,b \mid c\right) = \frac{p\left(a,b,c\right)}{p\left(c\right)}$$

(why?)

Case two where c is observed

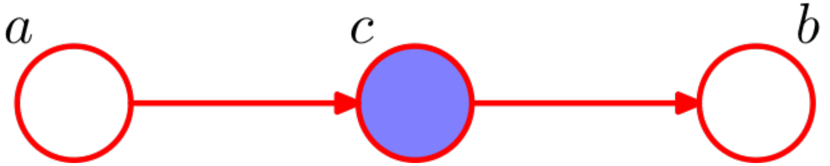


$$a \perp b \mid c$$

$$p(a,b \mid c) = \frac{p(a,b,c)}{p(c)} \qquad \text{(definition)}$$

$$= \frac{p(a)p(c|a)p(b|c)}{p(c)} \qquad \text{(why?)}$$

Case two where c is observed



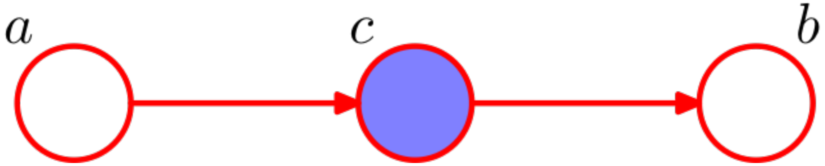
$$\mathrm{a} \perp \mathrm{b} \mid \mathrm{c}$$

$$p(a,b \mid c) = \frac{p(a,b,c)}{p(c)} \qquad \text{(definition)}$$

$$= \frac{p(a)\,p(c|a)\,p(b|c)}{p(c)} \qquad \text{(from graph)}$$

$$= \frac{p(a)\,p(a|c)\,p(c)\,p(b|c)}{p(a)\,p(c)} \qquad \text{(why?)}$$

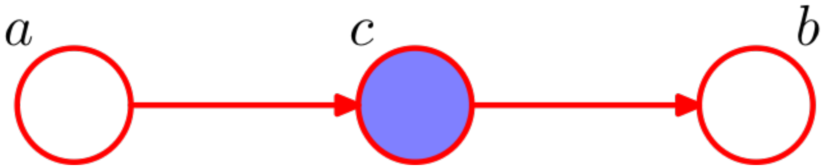
Case two where c is observed



$$a \perp b \mid c$$

$$
\begin{aligned}
p(a,b \mid c) &= \frac{p(a,b,c)}{p(c)} && \text{(definition)} \\
&= \frac{p(a)\,p(c|a)\,p(b|c)}{p(c)} && \text{(from graph)} \\
&= \frac{p(a)\,p(a|c)\,p(c)\,p(b|c)}{p(a)\,p(c)} && \text{(Bayes on } p(c|a)\text{)} \\
&= p(a|c)\,p(b|c) && \text{(why?)}
\end{aligned}
$$

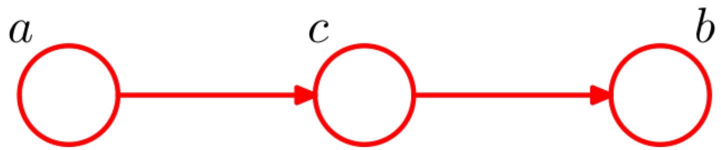
Case two where c is observed



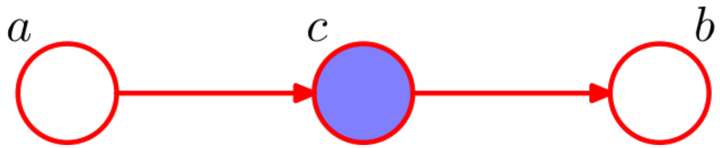
$a \perp b \mid c$ ✓

$$
\begin{aligned}
p(a,b \mid c) &= \frac{p(a,b,c)}{p(c)} && \text{(definition)} \\
&= \frac{p(a)\,p(c|a)\,p(b|c)}{p(c)} && \text{(from graph)} \\
&= \frac{p(a)\,p(a|c)\,p(c)\,p(b|c)}{p(a)\,p(c)} && \text{(Bayes on } p(c|a)\text{)} \\
&= p(a|c)\,p(b|c) && \text{(canceling factors)}
\end{aligned}
$$

Case two (head-to-tail) summary



$a \not\perp b$



$a \perp b \mid c$

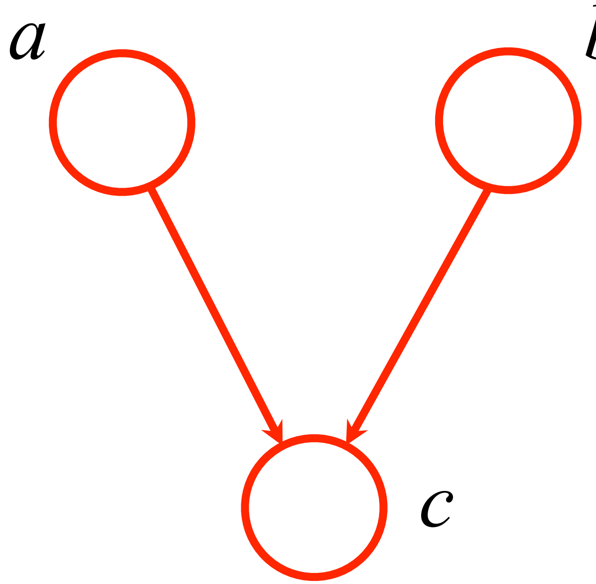
Head-to-tail case

With no conditioning = no independence
With conditioning = independence

(Same as tail-to-tail case!)

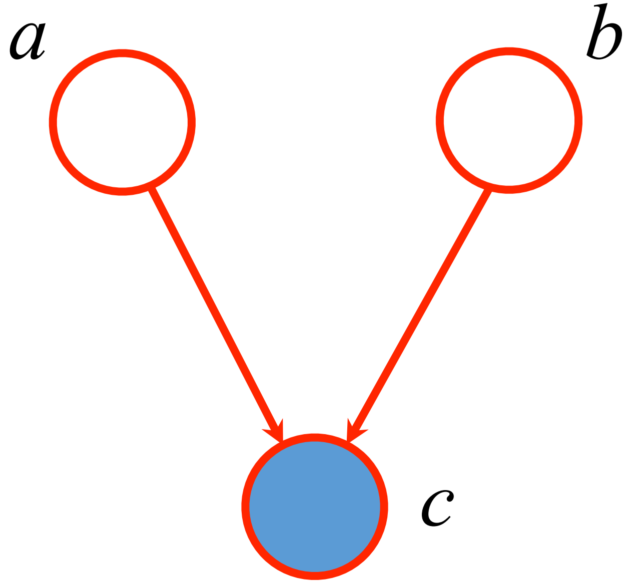
Case three (head-to-head)

Are a and b independent $(a \perp b)$*?*



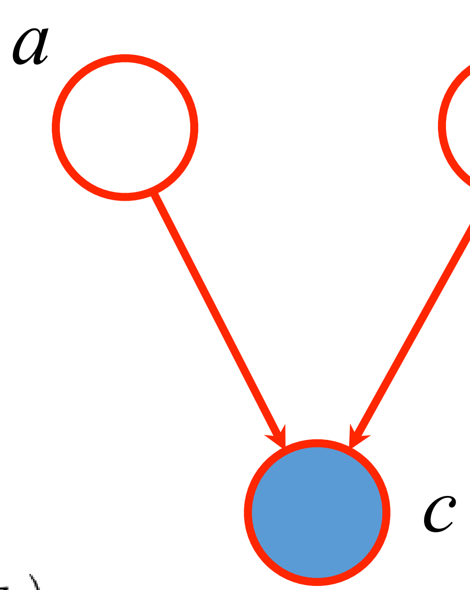
$$p(a, b) = \sum_c p(a)p(b)p(c \mid a, b) = p(a)p(b)$$

Are a and b conditionally independent ($a \perp b \mid c$)?



$$\mathbf{p(a,b,c) = p(a)p(b)p(c|a,b)}$$

Are a and b conditionally independent ($a \perp b \mid c$)?



$$p(a,b|c) = \frac{p(a,b,c)}{p(c)}$$

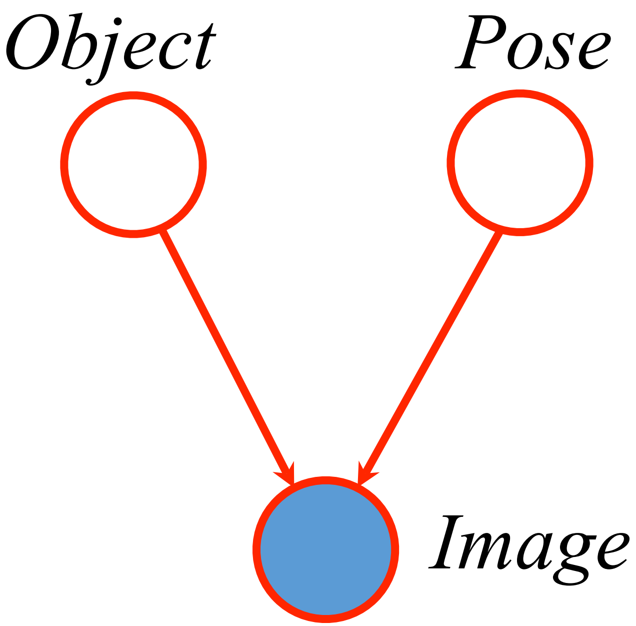
$$= \frac{p(a)p(b)p(c|a,b)}{p(c)}$$

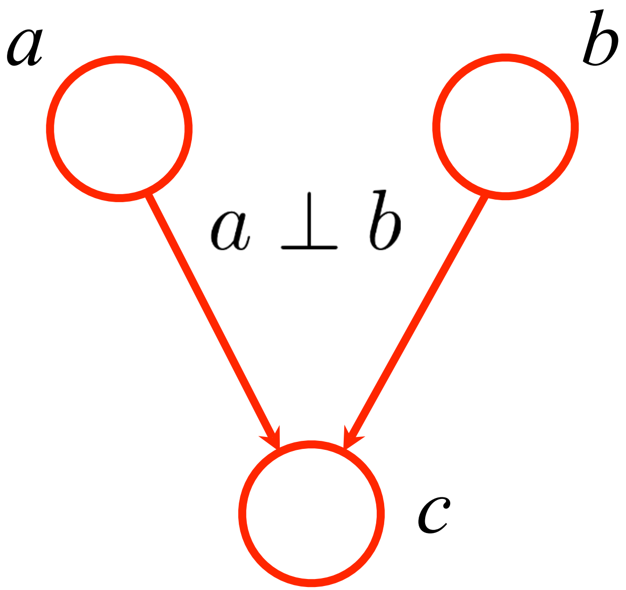
$$\neq p(a|c)p(b|c) \quad \text{(in general)}$$

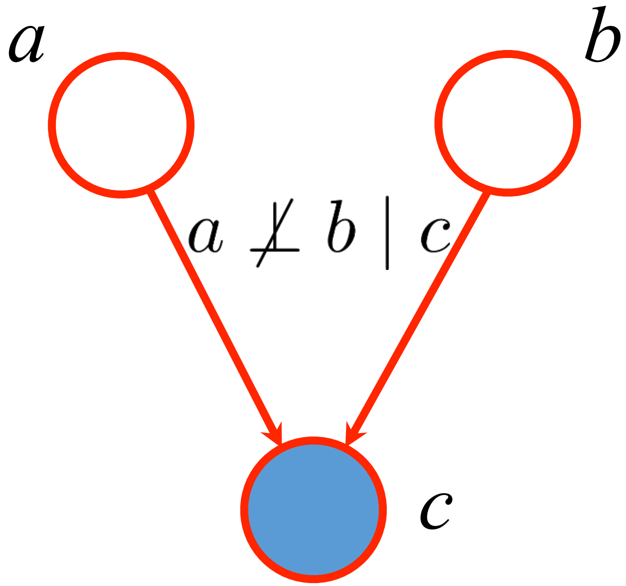
Attempt at algebraic proof.

Unless the algebra reduces to something obviously false, we typically look for a counter example

Both latent variables must explain same observed data so become dependent







Phenomenon in Bayes networks known as **explaining away**

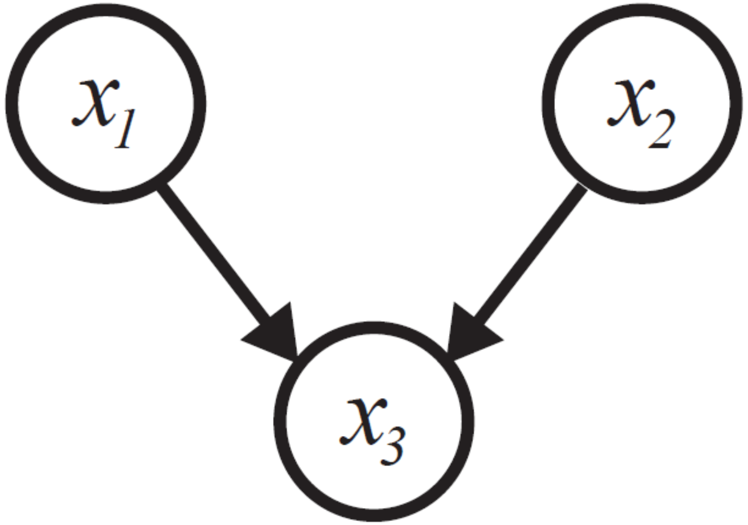
Markov Properties

*How can we be sure a PGM is **correct** for a distribution p(x)?*

Probability Model:

$$p(x_1, x_2, x_3) = p(x_1)p(x_2)p(x_3 \mid x_1, x_2)$$

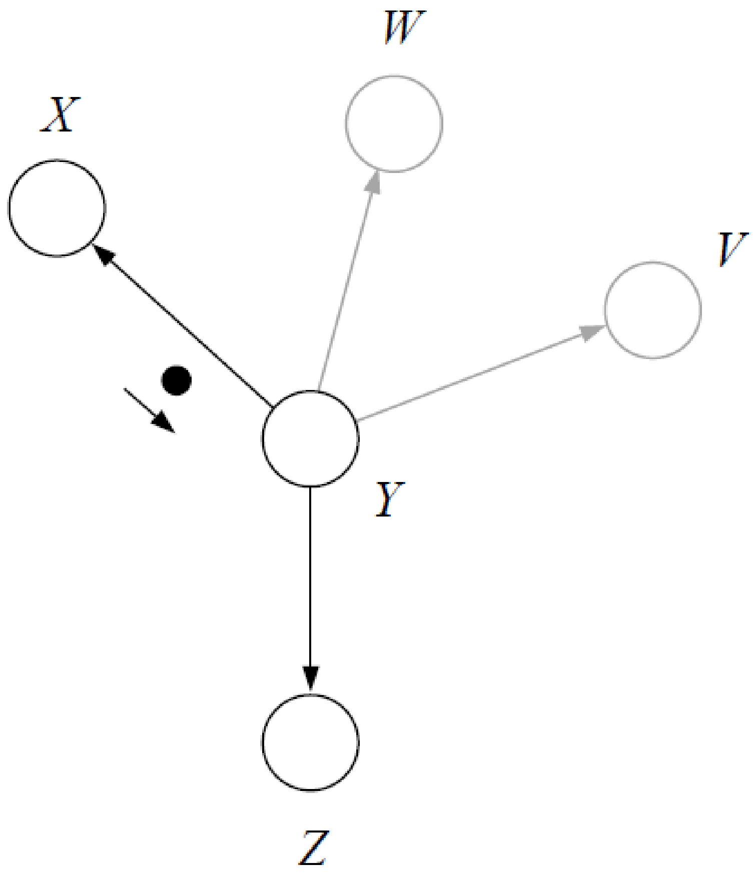
Graphical Model:



It must satisfy **all** of the conditional independencies of p(x), then we say p(x) **is Markov with respect to** the graph.

Bayes Ball Algorithm

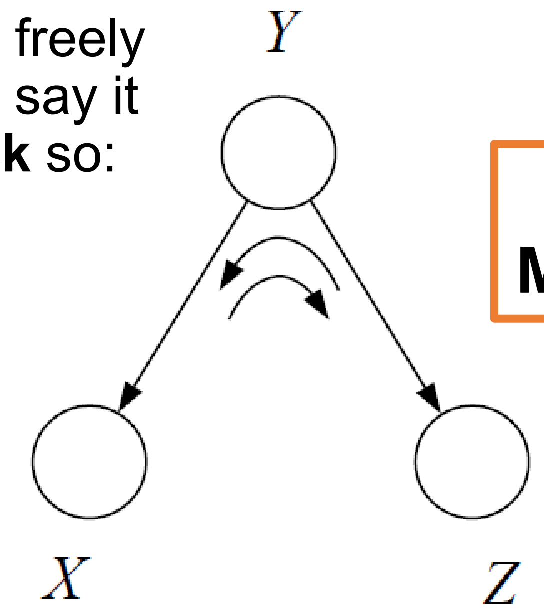
To test if $X \perp Z \mid Y$ imagine rolling a "ball" from X towards Z



The ball follows rules defined by the canonical 3-node subgraphs we've discussed

The ball passes freely from X-to-Z, we say it **does not block** so:

$$X \not\perp Z$$

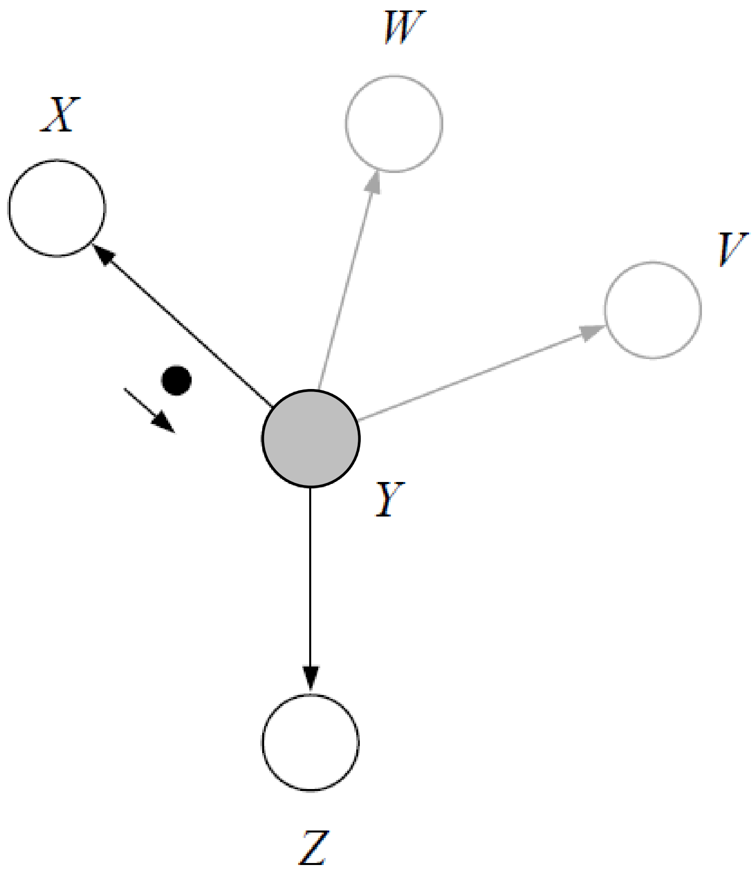


Reading: Murphy Sec. 10.5

[Images: Michael I Jordan (unpublished)]

Bayes Ball Algorithm

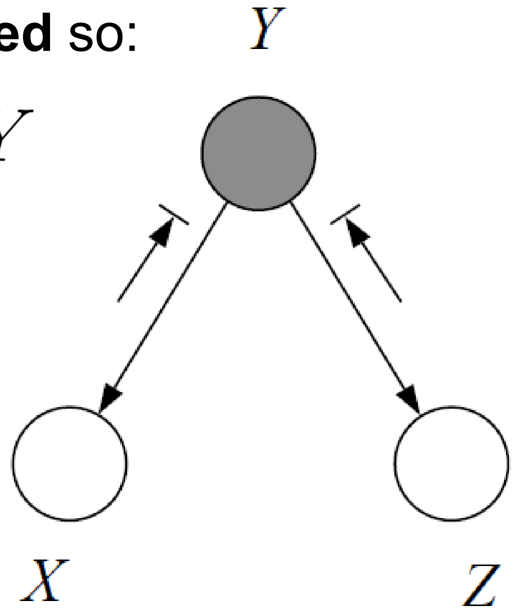
To test if $X \perp Z \mid Y$ imagine rolling a “ball” from X towards Z



The ball follows rules defined by the canonical 3-node subgraphs we’ve discussed

The ball **is blocked** so:

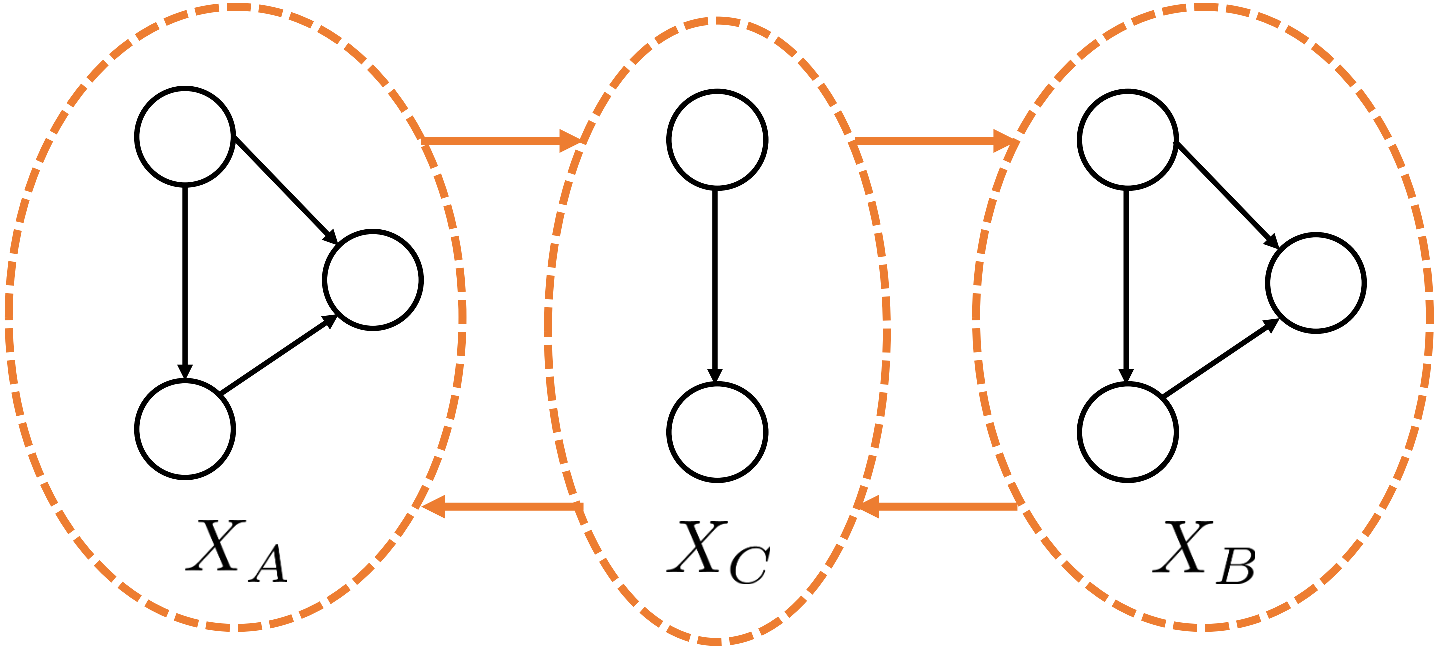
$$X \perp Z \mid Y$$



[Images: Michael I Jordan (unpublished)]

Directed Separation (d-Separation)

To test if $X_A \perp X_B \mid X_C$ roll ball from *every node in* X_A …



If *any* ball reaches *any node* in X_B then

$X_A \not\perp X_B \mid X_C$

Otherwise:

$X_A \perp X_B \mid X_C$

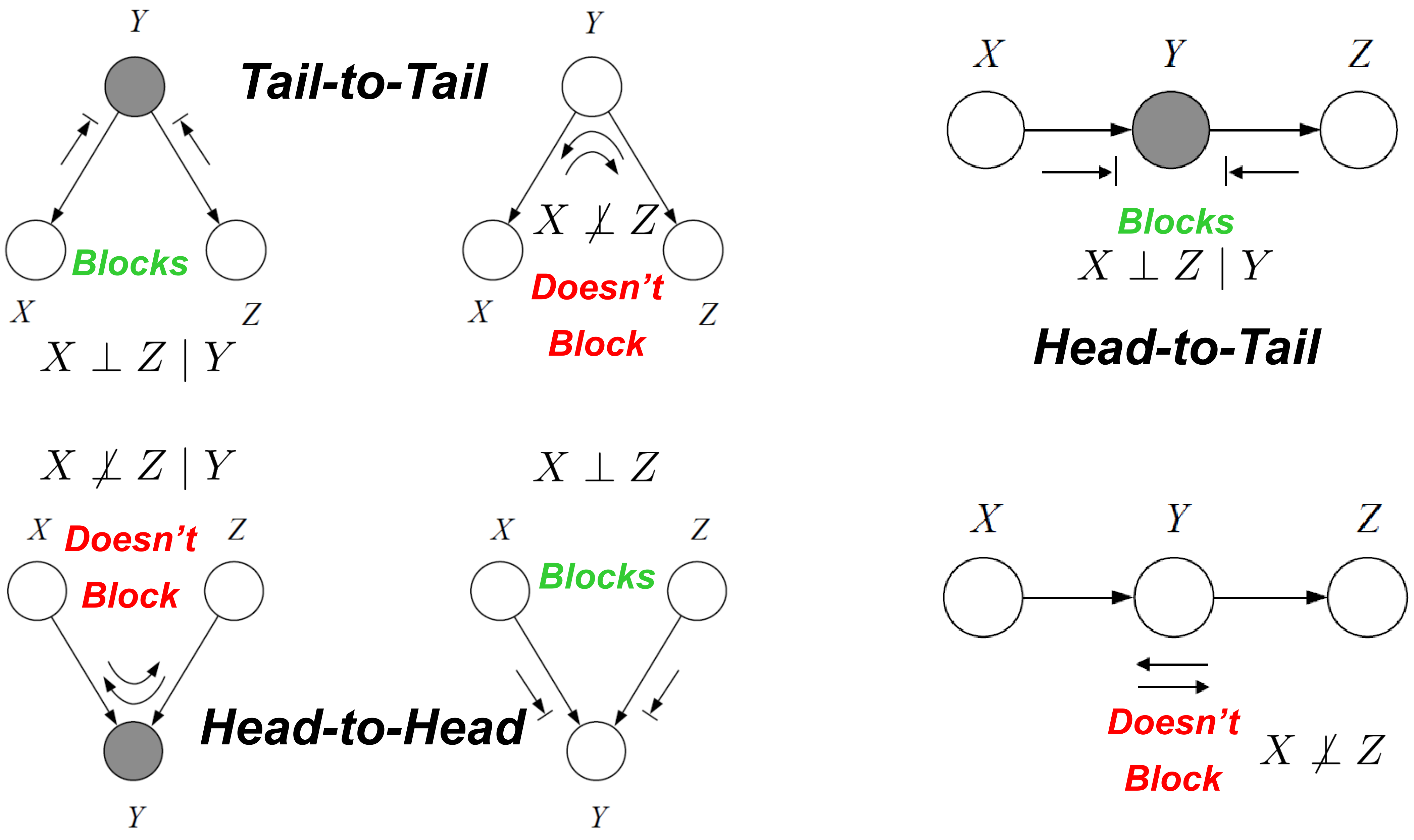
Tests for property of ***directed separation*** **(d-separation)**: if X_C separates / blocks X_A from X_B then $X_A \perp X_B \mid X_C$

Administrative Items

- HW2 Due Tonight

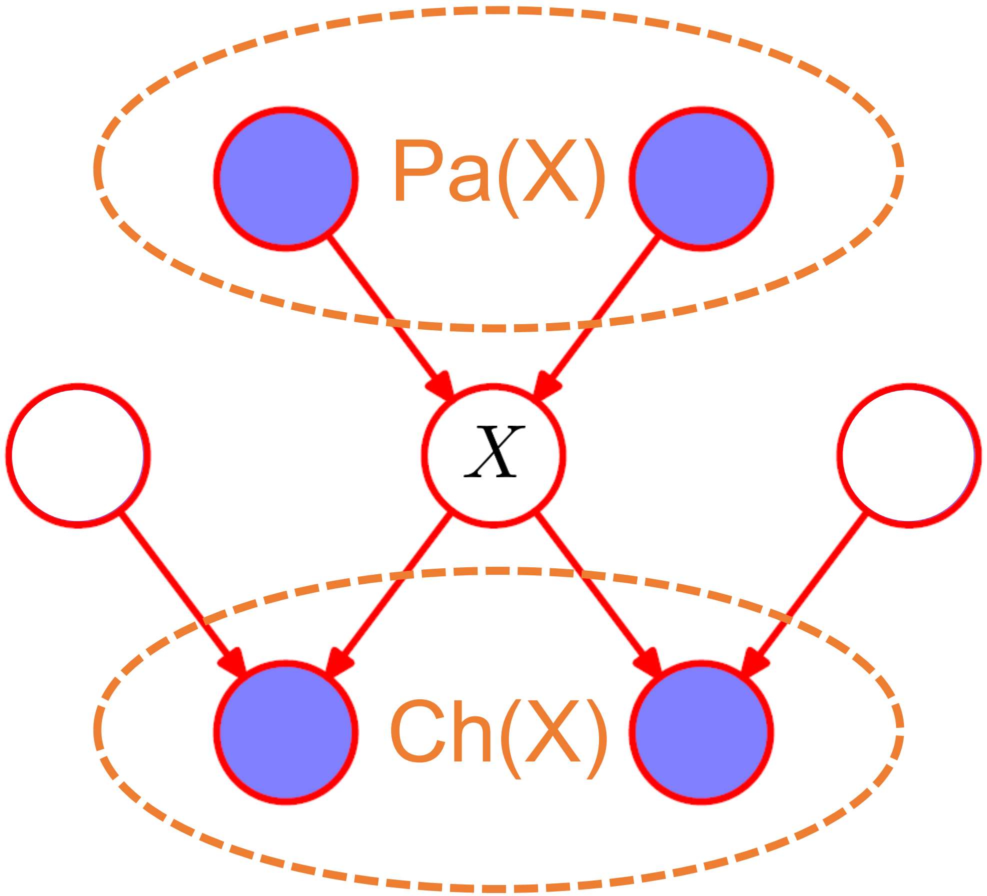
- HW3 Out Tonight
  - Due Wednesday, 2/16
  - Bayesian inference and Bayes Nets
  - 4 Questions
  - 6 Points

Bayes Ball Algorithm



Markov Blanket

Question Is X conditionally independent of all other nodes in graph given its **parents** and **children?**

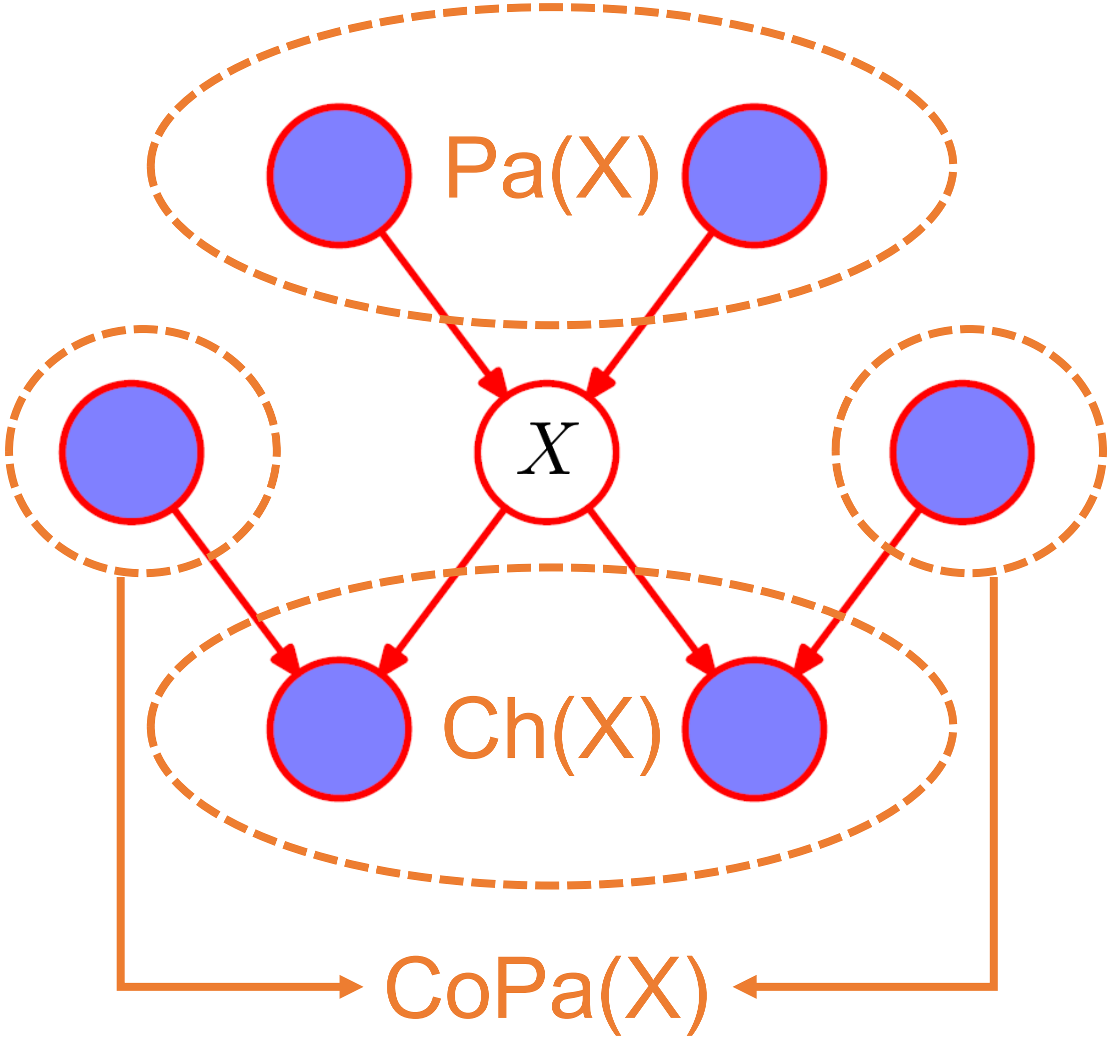


Markov Blanket

Question Is X conditionally independent of all other nodes in graph given its **parents** and **children?**

Answer No. It still depends on **co-parents.**

WHY?



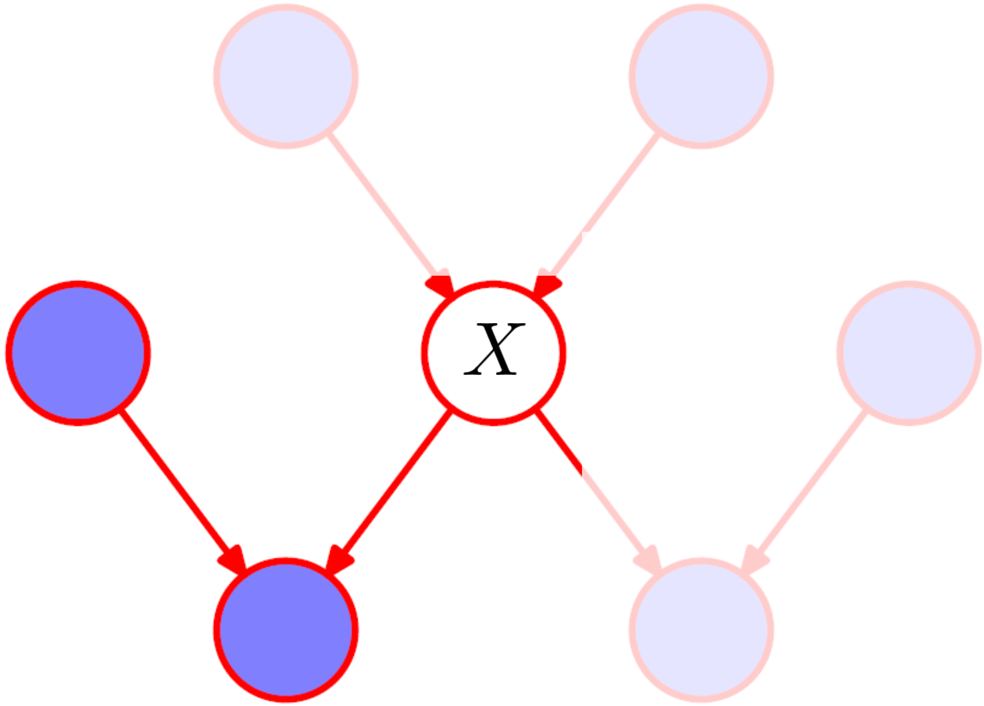
Markov Blanket

Question Is X conditionally independent of all other nodes in graph given its **parents** and **children?**

Answer No. It still depends on **co-parents.**

WHY?

Head-to-head conditional dependence from *explaining away property*



We refer to this conditioning set as the *Markov Blanket* of X…

Markov Blanket

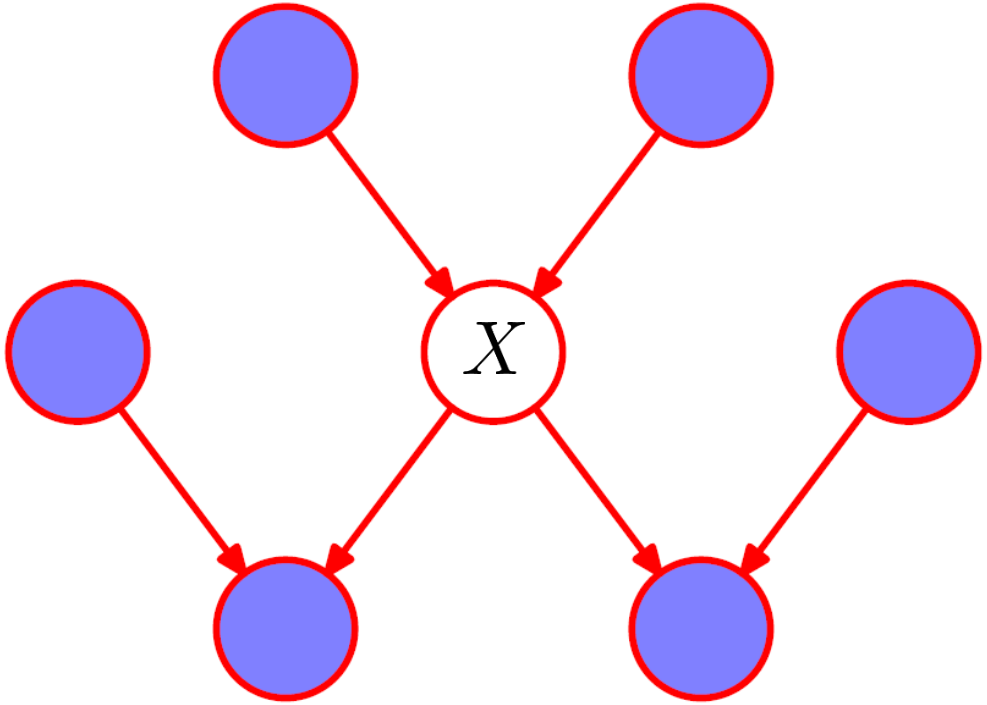
X conditionally independent of all other nodes, given its Markov blanket

Definition *A RV X with distribution p(x) that is Markov w.r.t. Bayes Net with graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has a* **Markov blanket** *given by:*

$$\mathrm{Mb}(X) = \mathrm{Pa}(X) \cup \mathrm{Ch}(X) \cup \mathrm{CoPa}(X)$$

For any $Y \notin \mathrm{Mb}(X)$:

$$X \perp Y \mid \mathrm{Mb}(X)$$



Markov blanket used to simplify inference and distribute computation (e.g. Gibbs sampler, variational inference, etc.)

Outline

Directed graphical models

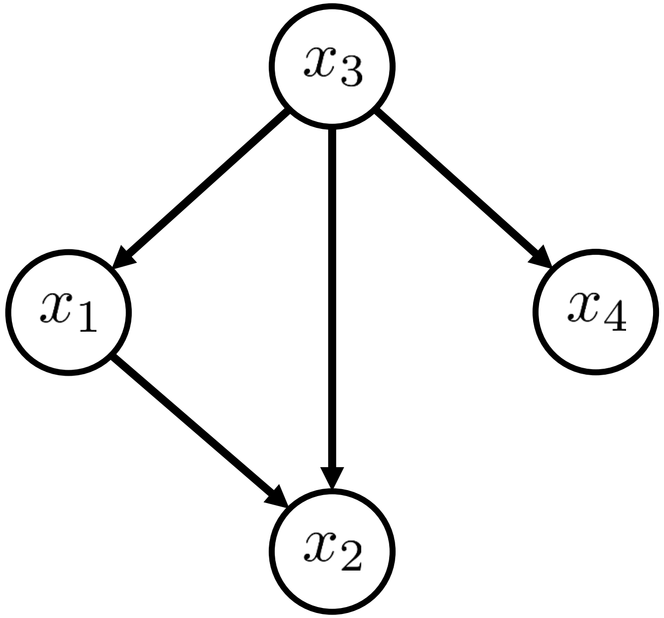
- Bayes Nets
- Conditional dependence

Undirected graphical models

- Markov random fields (MRFs)
- Factor graphs

Directed PGM = Bayes Network

Model factors are normalized conditional distributions:



$$p(x) = \prod_{s \in \mathcal{V}} p(x_s \mid x_{\mathrm{Pa}(s)})$$

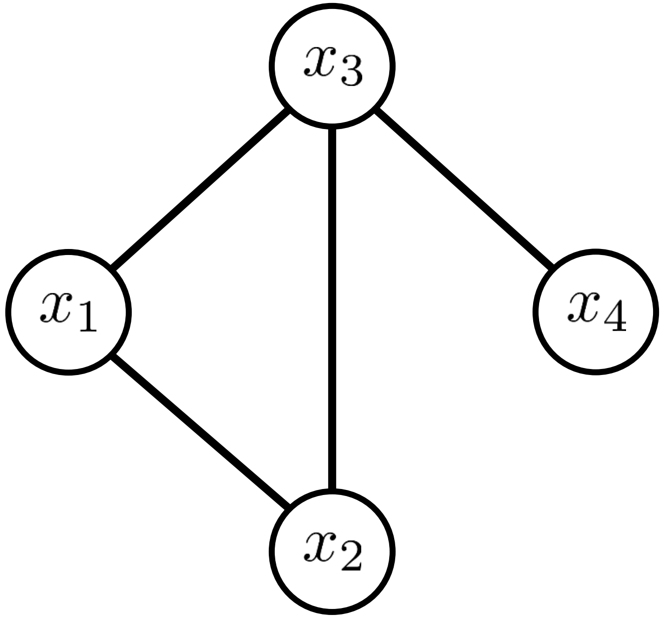
Parents of node *s*

Locally normalized factors yield *globally normalized* joint probability

Often difficult to specify joint in terms of product of normalized probabilities…

Markov Random Field

Specify joint as product of unnormalized functions…



Functions model how variables interact

$$p(x) = \frac{1}{Z}\psi_a(x_1, x_2, x_3)\psi_b(x_3, x_4)$$

Global normalization constant

Potential functions ψ and are non-negative and their product is normalizable…**they are not unnormalized probabilities!**

- More general class of models than Bayes Nets
- Any Bayes Net easily convertes to MRF by dropping local normalizers
- MRF→Bayes Net not straightforward

Factorized Probability Distributions

A probability distribution over RVs $x = (x_1, \ldots, x_d)$ can be written as a product of factors,

$$p(x) = \frac{1}{Z} \prod_{c \in \mathcal{C}} \psi_c(x_c)$$

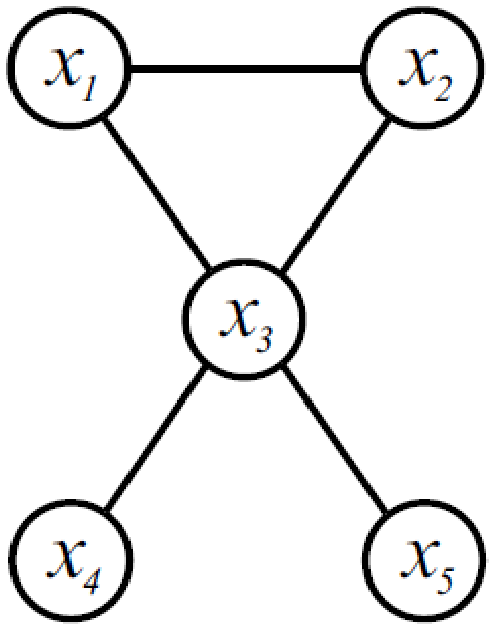
<u>Where:</u>

- $\mathcal{C}$ a collection of subsets of indices $\{1, \ldots, d\}$
- $\psi(\cdot)$ are nonnegative *factors* (or *potential functions*)
- Z the normalizing constant (or *partition function*)

$$Z = \int \prod_{c \in \mathcal{C}} \psi_c(x_c) \, dx_c$$

Undirected Graphical Models

A **graph** $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a set of vertices $\mathcal{V}$ and edges $\mathcal{E}$. An edge $(s,t) \in \mathcal{E}$ connects two vertices $s,t \in \mathcal{V}$.



In **undirected models** edges are specified irrespective of node ordering so that,

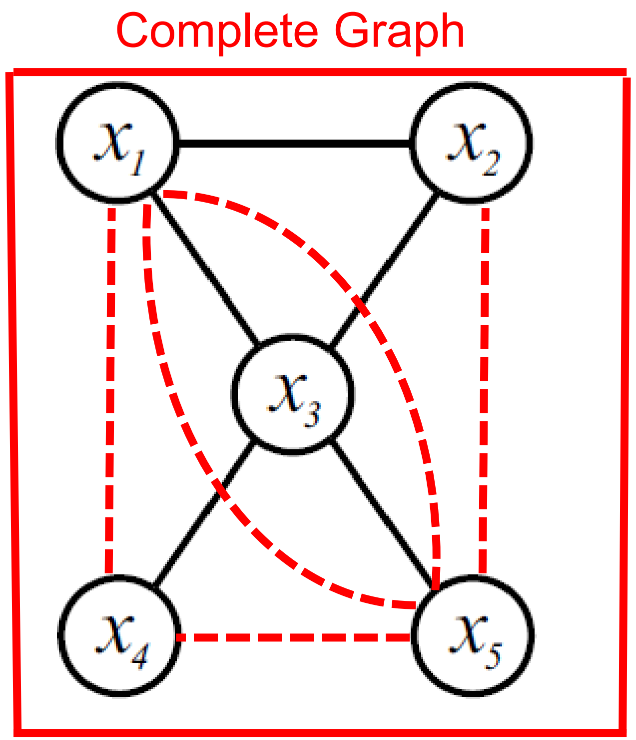
$$(s,t) \in \mathcal{E} \Leftrightarrow (t,s) \in \mathcal{E}$$

Distributions are typically specified with unknown normalization (easier to specify),

$$p(x) \propto \prod_{c \in \mathcal{C}} \psi_c(x_c)$$

Markov Random Fields (MRFs)

A factor $\psi_c(x_c)$ corresponds to a clique $c \in \mathcal{C}$ (fully connected subgraph) in the MRF



An MRF does not imply a unique factorization, for example all the following are "*valid*":

$$\psi(x_1, x_2, x_3, x_4, x_5)$$

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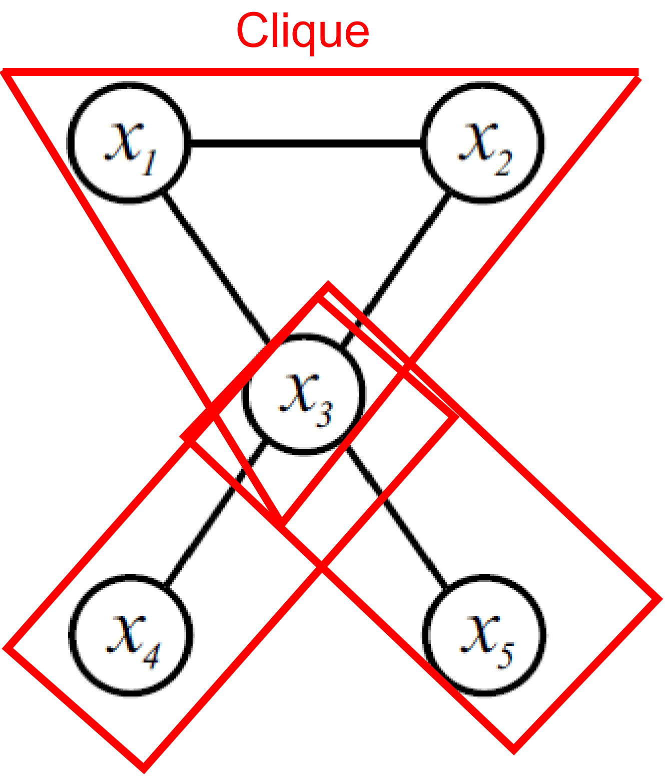

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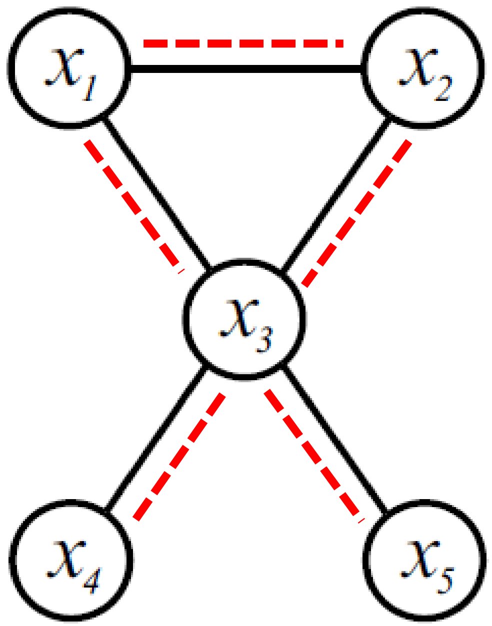
$$\psi(x_1, x_2, x_3, x_4, x_5)$$

$$\psi(x_1, x_2, x_3)\psi(x_3, x_4)\psi(x_3, x_5)$$

Markov Random Fields (MRFs)

A factor $\psi_c(x_c)$ corresponds to a clique $c \in \mathcal{C}$ (fully connected subgraph) in the MRF

Pairwise MRF



An MRF does not imply a unique factorization, for example all the following are "*valid*":

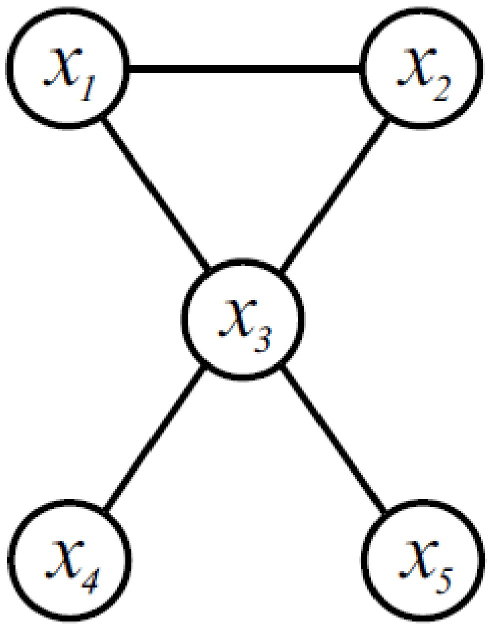
$$\psi(x_1, x_2, x_3, x_4, x_5)$$

$$\psi(x_1, x_2, x_3)\psi(x_3, x_4)\psi(x_3, x_5)$$

$$\psi(x_1, x_2)\psi(x_2, x_3)\psi(x_1, x_3)\psi(x_3, x_4)\psi(x_3, x_5)$$

A **minimal factorization** is one where all factors are **maximal cliques** (not a strict subset of any other clique) in the MRF

Example



Interaction potential between each pair of nodes $(i,j) \in \mathcal{E}$ is exponentiated quadratic,

$$\psi_{ij}(x_i, x_j) = \exp\left(-\frac{1}{2}(x_i - x_j)^2\right)$$

Joint probability is proportional to product,

$$p(x) = \frac{1}{Z}\psi_{12}(x_1, x_2)\psi_{13}(x_1, x_3)\psi_{23}(x_2, x_3)\psi_{34}(x_3, x_4)\psi_{35}(x_3, x_5)$$

Question What named distribution is p(x)?

Answer Multivariate Gaussian

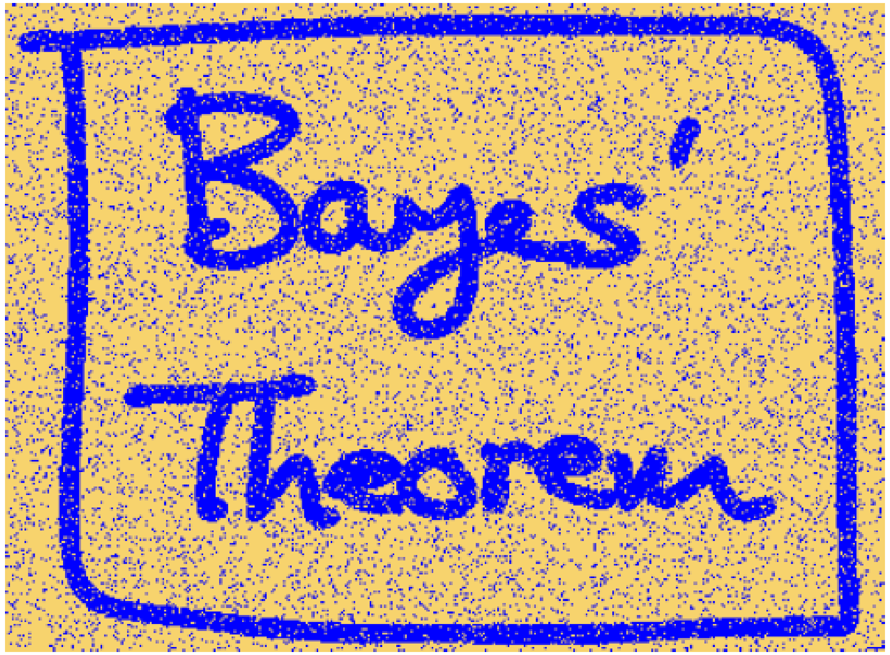
$$p(x) = \mathcal{N}(x \mid \mu, \Sigma) \qquad Z = (2\pi)^{5/2}|\Sigma|^{1/2}$$

Can easily read off inverse covariance…

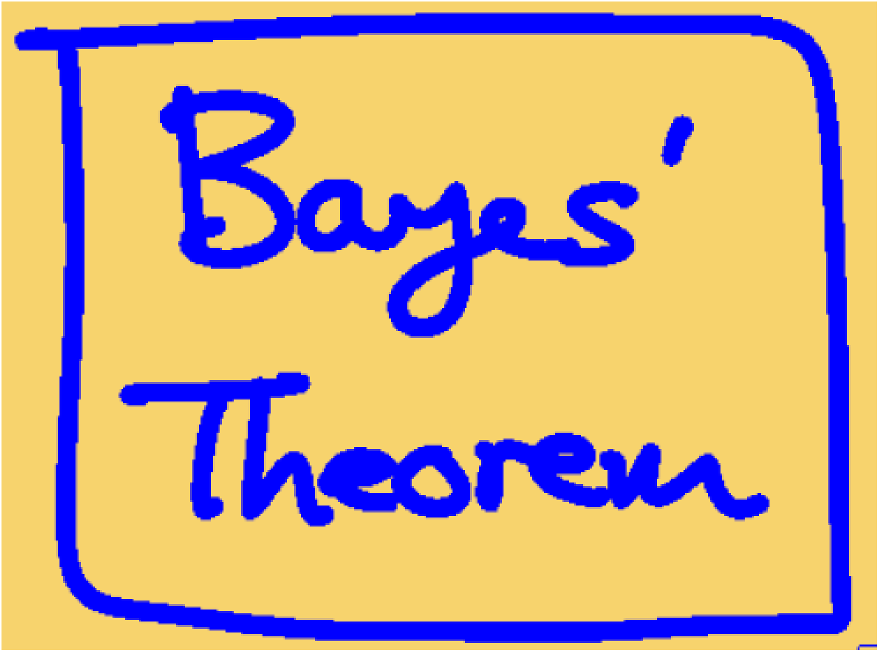
$$\Sigma^{-1} = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 4 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Example: Image Denoising

Noisy Image



Latent Image



Problem Given observed image corrupted by i.i.d. noise, infer "clean" denoised image.

[Source: Bishop, C. PRML]

Example: Image Denoising

Model Assume binary image with latent pixels $x_i \in \{0,1\}$ and observed $y_i \in \{0,1\}$.

Observed pixels randomly flipped i.i.d.,

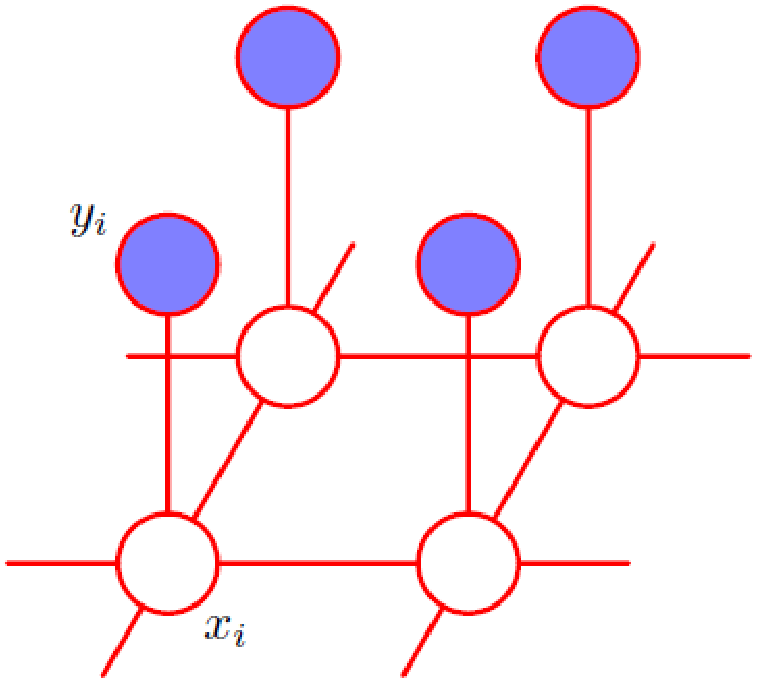
$$\log \phi_i(x_i) = \eta x_i y_i$$

Eta parameter controls noise

Neighboring pixels should appear similar,

$$\log \phi_{ij}(x_i, x_j) = \beta x_i x_j$$

Beta parameter controls *smoothness*



Full MRF (in “energy” form):

$$E(x,y) = -\sum_i \log \phi_i(x_i) - \sum_{(i,j)} \log \phi_{ij}(x_i, x_j)$$

Often specify MRF in “energy” or negative log-probability form (minimize energy → maximize probability)

[Source: Bishop, C. PRML]

Normalizing MRFs

Joint probability of *image denoising* model,

$$p(x,y) = \frac{1}{Z} \exp\{-E(x,y)\}$$

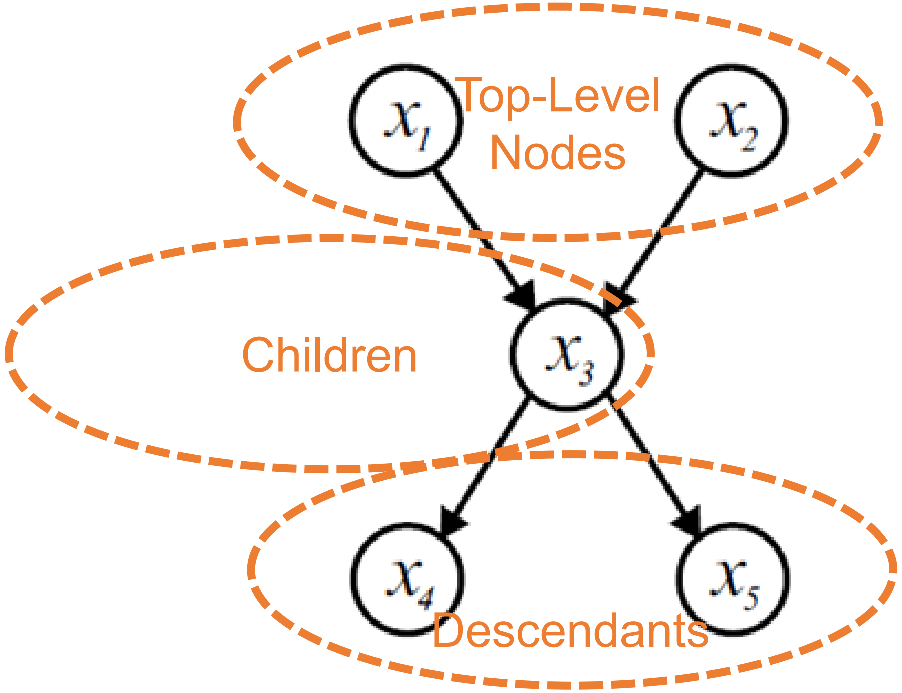
Normalization (a.k.a. partition function) for N pixel image:

$$Z = \underbrace{\sum_{x_1} \sum_{x_2} \cdots \sum_{x_N}}_{O(2^N) \text{ terms}} \exp\{-E(x,y)\}$$

Normalization not always possible in closed-form : i.e. need to sum over *all possible N-pixel images*

Often ignore Z and specify MRFs up to proportionality…

Simulation



Bayes Nets Straightforward simulation via ancestral sampling successively samples from conditionals:

$$p(\mathbf{x}) = \prod_{i \in \mathcal{V}} p(x_i \mid x_{\mathrm{Pa}(i)})$$

so

$$x_i \sim p(x_i \mid x_{\mathrm{Pa}(i)})$$

Undirected Graphs Sampling not as straightforward…

- Lack locally normalized conditionals to sample from
- Lack partial ordering of nodes

We will return to this when we discuss Markov chain Monte Carlo

Conditional Independence (Undirected)

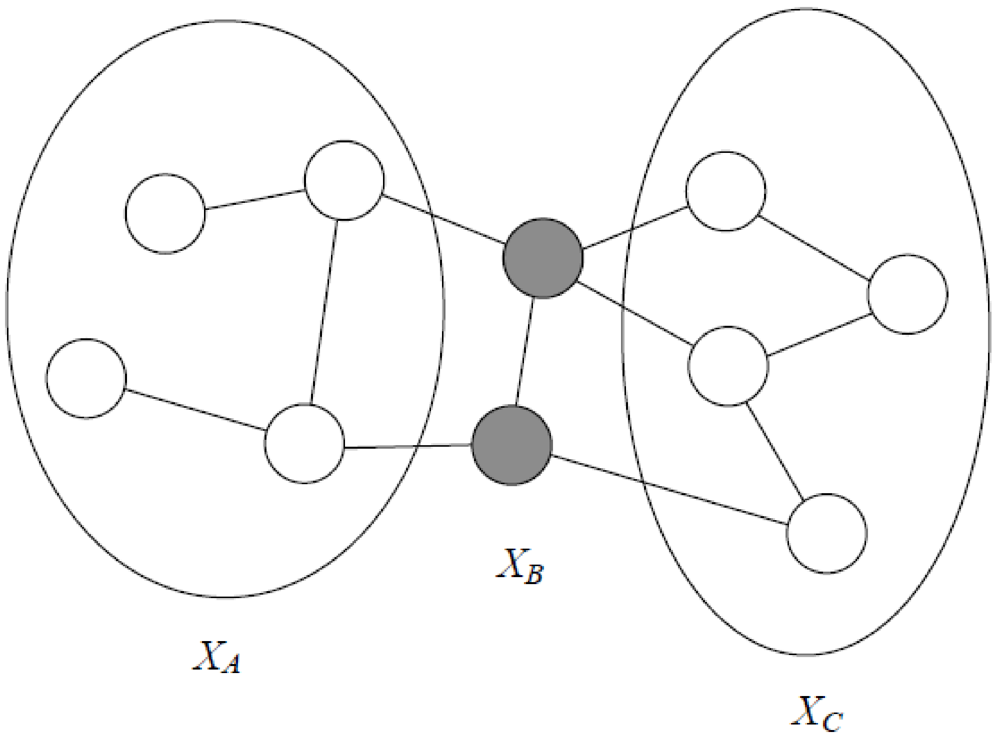
We say x_A and x_C are *independent* or $x_A \perp\!\!\!\perp x_C$ if:

$$p(x_A, x_C) = p(x_A)p(x_C)$$

We say they are *conditionally independent* or $x_A \perp\!\!\!\perp x_C \mid x_B$ if:

$$p(x_A, x_C \mid x_B) = p(x_A \mid x_B)p(x_C \mid x_B)$$

Def. We say $p(x)$ is *globally Markov* w.r.t. $\mathcal{G}$ if $x_A \perp\!\!\!\perp x_C \mid x_B$ in any separating set of $\mathcal{G}$.



Conditional independence in undirected graphical models is defined by separating sets

[Source: Michael I. Jordan]

Global & Local Markov Properties

Global Markov Property

- Set B separates A from C if all paths from A to C pass through B
- By definition, distribution is Markov if and only if for any B separating A and C:

$$p(x_A, x_C \mid x_B) = p(x_A \mid x_B)p(x_C \mid x_B)$$

$$p(x_A \mid x_B, x_C) = p(x_A \mid x_B) \qquad p(x_C \mid x_B, x_A) = p(x_C \mid x_B)$$

Local Markov Property

- Given its *neighbors*, each node is independent of all other variables

$$p(x_s \mid x_{\mathcal{V} \setminus s}) = p(x_s \mid x_{\Gamma(s)})$$

$$\Gamma(s) = \{t \in \mathcal{V} \mid (s,t) \in \mathcal{E}\}$$

Markov blanket only includes immediate neighbors (we needed co-parents in Bayes nets)

- This local Markov property is a special case of the global Markov property

[Source: Erik Sudderth]

Hammersley-Clifford Theorem

Thorem (Hammersley-Clifford). *Let $\mathcal{C}$ denote the set of cliques of an undirected graph $\mathcal{G}$. A probability distribution defined as a normalized product of non-negative potential functions on those cliques is then always Markov with respect to $\mathcal{G}$:*

$$p(x) \propto \prod_{c \in \mathcal{C}} \psi_c(x_c)$$

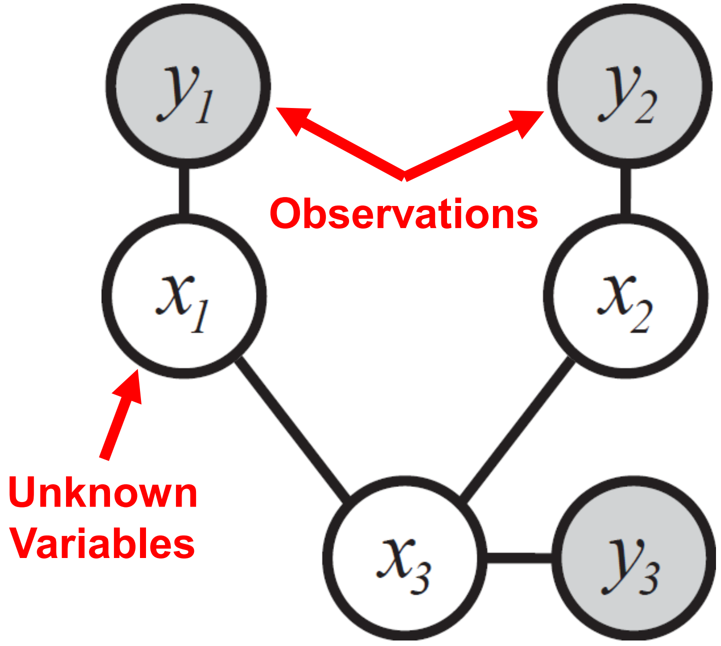
Conversely, any strictly positive density which is Markov with respect to $\mathcal{G}$ can be represented in this factored form.

Global Markov Property ⟷ **Joint Factorization**

(Graph Separation Implies Independence) (Potential Function for Each Clique)

Pairwise Markov Random Field

Often easier to specify and do inference on pairwise model



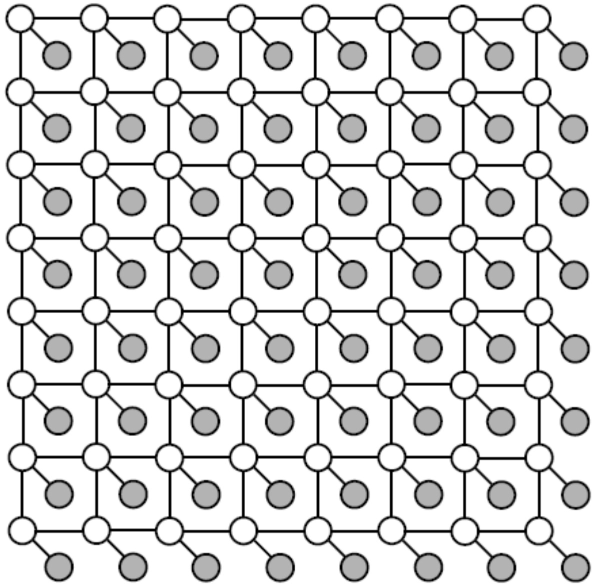
$$p(x, y) \propto \prod_{s \in \mathcal{V}} \underbrace{\psi_s(x_s, y)}_{\text{Likelihood}} \prod_{(s,t) \in \mathcal{E}} \underbrace{\psi_{st}(x_s, x_t)}_{\text{Prior}}$$

Restricted class of MRFs

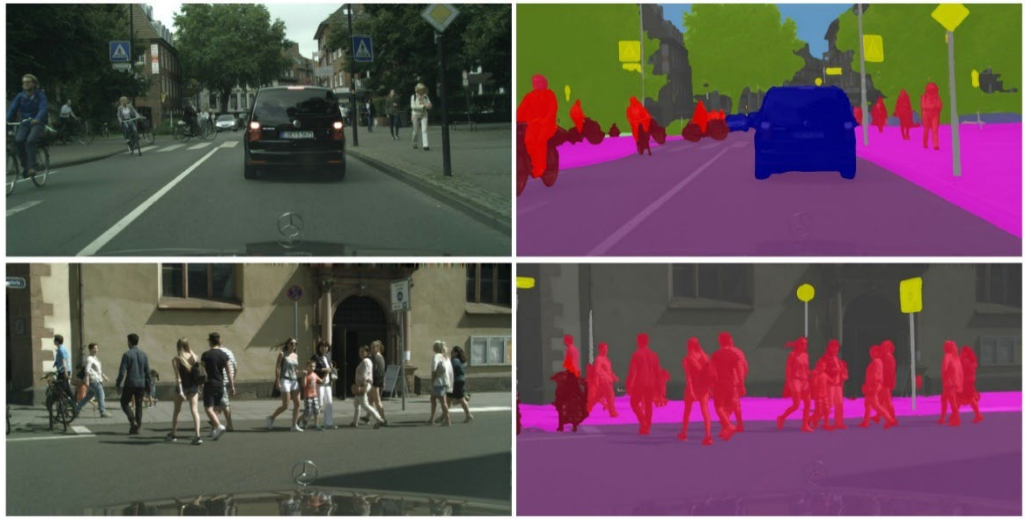
- 2-node factor exists for every edge
- Explicit factorization of joint distribution
- High-order factors not always easily decomposed into pairwise terms

Example: Image Segmentation

[Source: Kundu, A. et al., CVPR16]



Don't need to know log-partition to specify model



Pairwise MRF *energy*: $-\log p(x,y) = \log Z + \sum_i \psi_i(x_i, y_i) + \sum_{(i,j)} \psi_{i,j}(x_i, x_j)$

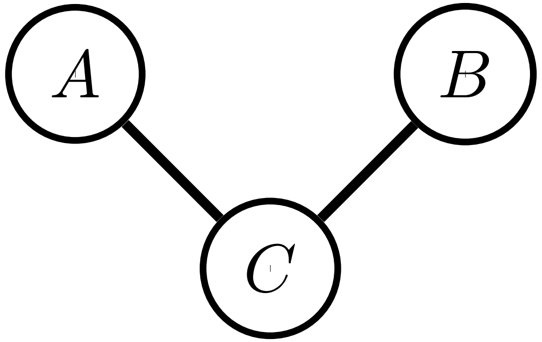
Don't need to specify normalized conditionals as in Bayes Nets

Low energy configurations = High probability

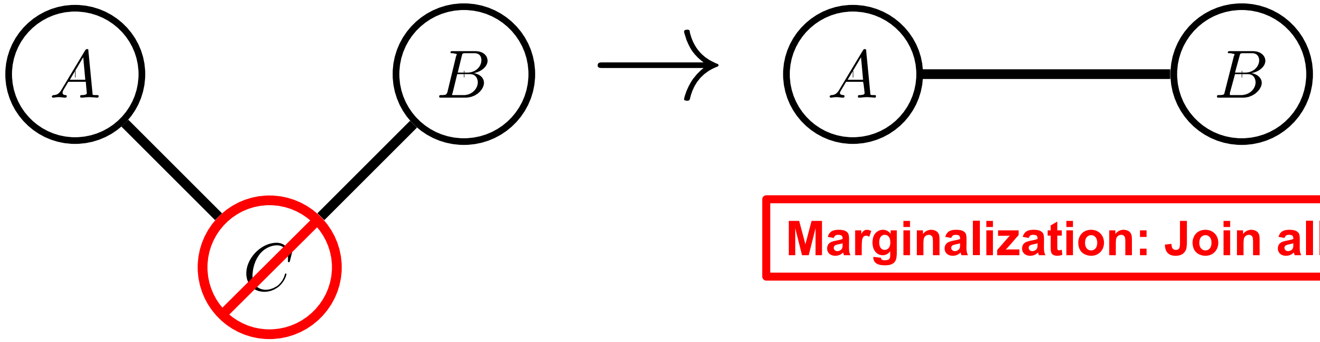
L2 Likelihood: $\psi_i(x_i, y_i) = \|x_i - y_i\|^2$ **Potts model:** $\psi_{i,j}(x_i, x_j) = \mathbb{I}(x_i = x_j)$

MAP (minimum energy) configuration = Piecewise constant regions

Transformations of Undirected Models



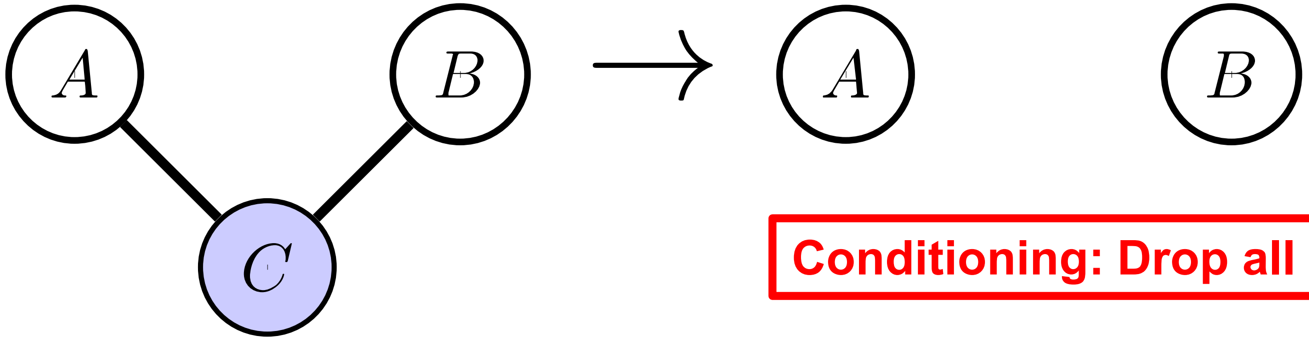
$$p(A, B, C) = \psi_{AC}(A, C)\psi_{BC}(B, C)/Z$$



$$p(A, B) \neq p(A)p(B)$$

Marginalization: Join all nodes that have path through C

Marginalising over C makes A and B (graphically) dependent.



$$p(A, B|C) = p(A|C)p(B|C)$$

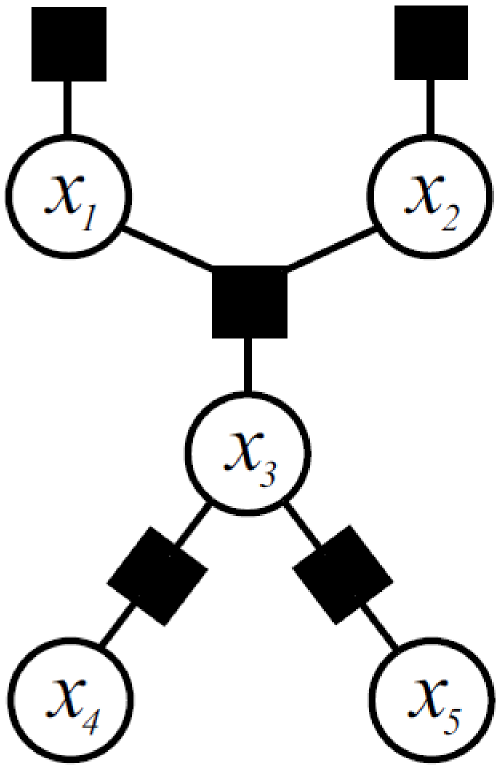
Conditioning: Drop all edges on path through C

Conditioning on C makes A and B independent:

[Source: Erik Sudderth]

Factor Graphs

A *hypergraph* $\mathcal{H} = (\mathcal{V}, \mathcal{F})$ where a *hyperedge* $f \in \mathcal{F}$ is a subset of vertices $f \subset \mathcal{V}$.



Factor node for each product term in the joint factorization:

Graphical model makes factorization explicit

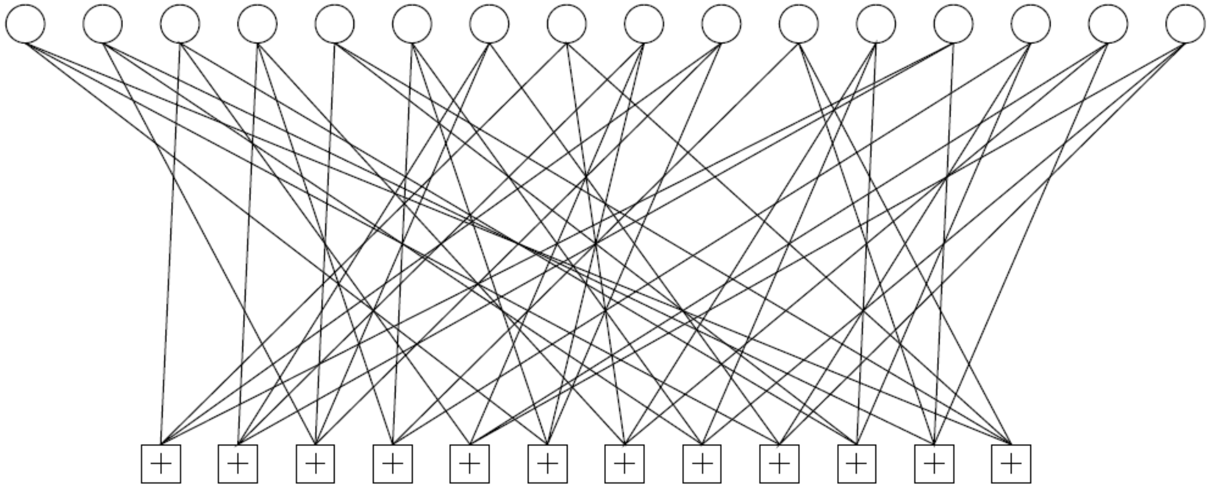
$$p(x) \propto \prod_{f \in \mathcal{F}} \psi_f(x_f)$$

where $x_f = \{x_i : i \in f\}$ the set of variables in factor *f*. For example:

$$\psi(x_1)\psi(x_2)\psi(x_1, x_2, x_3)\psi(x_3, x_4)\psi(x_3, x_5)$$

Example: Low Density Parity Check Codes

Factor Graph Representation



Problem Setup

- A code t is transmitted over a noisy
- Received code r is corrupted by noise
- Estimate the most probable code that was sent $t*$ (*maximum a posteriori)*

Transmitted Code

$t \sim p(t)$

Noisy Channel

Received Code

$r \mid t \sim p(r \mid t)$

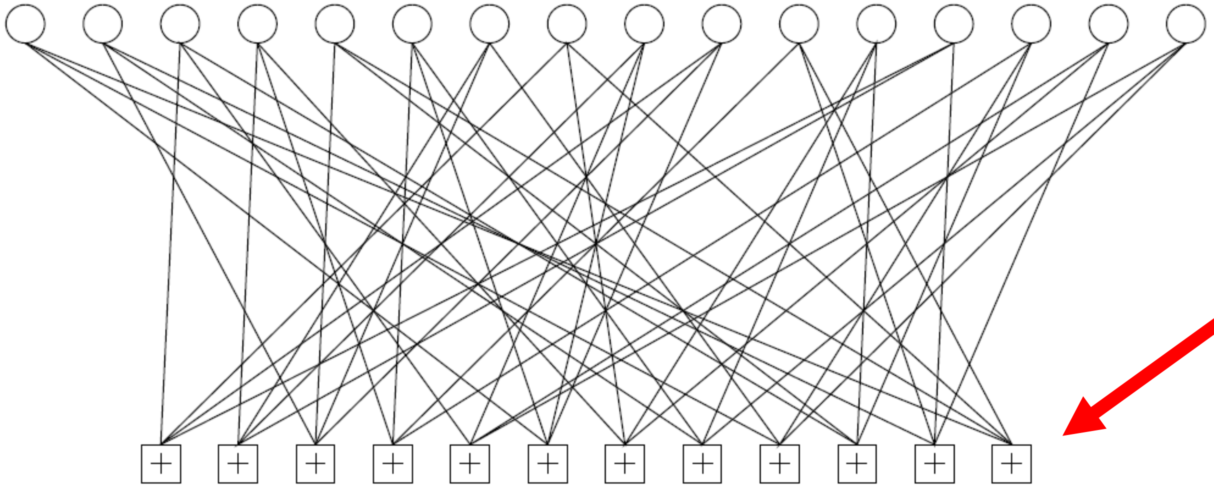
Decoder

$t^* = \arg\max_t p(t \mid r)$

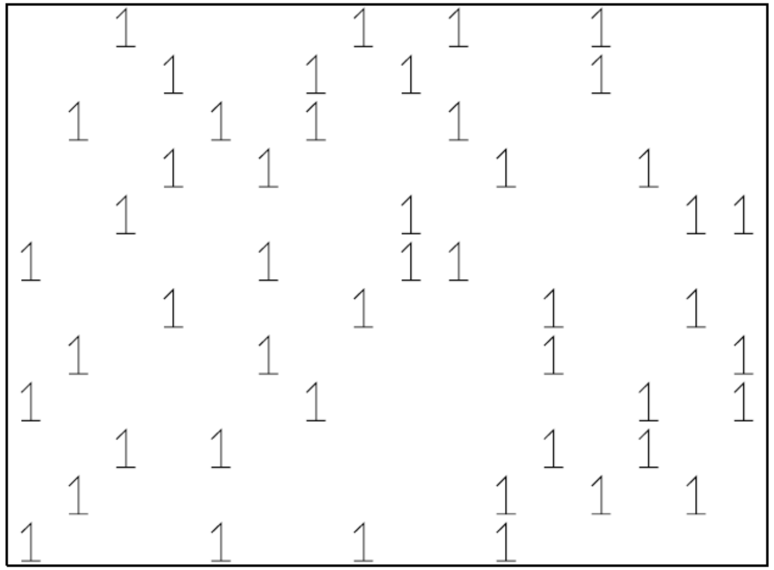
[Source: David MacKay]

Example: Low Density Parity Check Codes

Factor Graph Representation



Sparse Parity Check Matrix

$\mathbf{H} =$ 

- Valid codes have zero parity: $p(t) \propto \mathbb{I}(Ht = 0 \bmod 2)$
- Chanel noise model arbitrary, e.g. flip bits w/ ϵ probability:

$$p(r \mid t) = \prod_n p(r_n \mid t_n) = \prod_n (1 - \epsilon)^{\mathbb{I}(r_n = t_n)} \epsilon^{\mathbb{I}(r_n \neq t_n)}$$

n-th bit

[Source: David MacKay]

Recap: Directed Models

- Distribution factorized as product of conditionals via chain rule

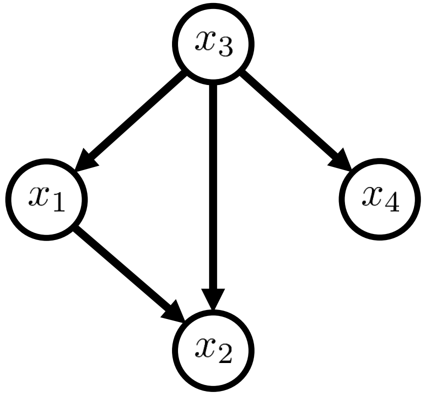
$$p(x_1, x_2, x_3, x_4) = p(x_3)p(x_1 \mid x_3)p(x_4 \mid x_1, x_3)p(x_2 \mid x_1, x_3, x_4)$$

- Choose ordering where terms simplify due to conditional independence

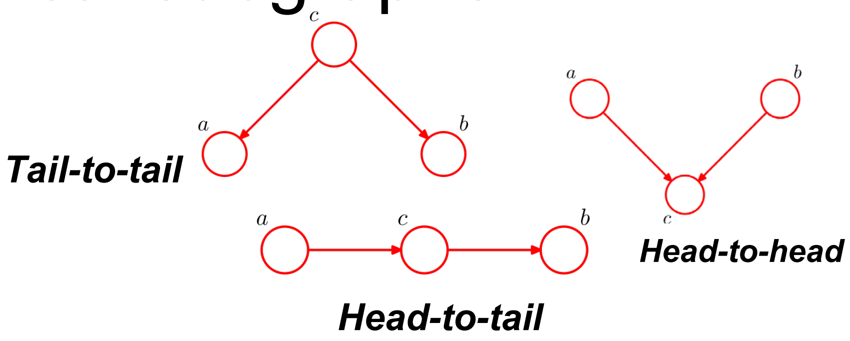
Eg. Suppose $x_4 \perp x_1 \mid x_3$ and $x_2 \perp x_4 \mid x_1$ then:

$$p(x) = p(x_3)p(x_1 \mid x_3)p(x_4 \mid x_3)p(x_2 \mid x_1, x_3)$$

- Directed graph encodes factorized distribution via conditional independence properties



- Test independence using canonical subgraphs:
- Straightforward simulation via **ancestral sampling**



Recap: Undirected Model

- Joint factorization as nonnegative factors (potentials) over subsets:

$$p(x) \propto \prod_{f \in \mathcal{F}} \psi_f(x_f)$$

- Easier to specify models compared to Bayes nets since:
  - Factors do not need to be normalized conditional probabilities
  - May specify up to unknown normalization constant
- Easier to verify Markov independence via *separating sets*
- Factorization ambiguous in MRFs, but explicit in factor graphs (factor graphs generally preferred)
- Simulation is not easy in general. Can't do ancestral sampling.